Universal coefficients for overconvergent cohomology and the geometry of eigenvarieties

David Hansen*
with an appendix by James Newton

October 2, 2012

Abstract

We prove a universal coefficients theorem for the overconvergent cohomology modules introduced by Ash and Stevens, and give several applications. In particular, we sketch a very simple construction of eigenvarieties using overconvergent cohomology and prove many instances of a conjecture of Urban on the dimensions of these spaces. For example, when the underlying reductive group is an inner form of GL_2 over a quadratic imaginary extension of $\mathbf{Q}$, the cuspidal component of the eigenvariety is a rigid analytic curve.

Contents

*Department of Mathematics, Boston College, Chestnut Hill MA; dvd.hnsn@gmail.com

1 Introduction

1.1 Background and results

Since the pioneering works of Serre [Ser73], Katz [Kat73], and especially Hida [Hid86, Hid88, Hid94] and Coleman [Col96, Col97], p-adic families of modular forms have become a major topic in modern number theory. Aside from their intrinsic beauty, these families have found applications towards Iwasawa theory, the Bloch-Kato conjecture, modularity lifting theorems, and the local and global Langlands correspondences [BC04, BC09, BC11, BLGGT12, Ski09, Wil90]. One of the guiding examples in the field is Coleman and Mazur's *eigencurve* [CM98, Buz07], a universal object parametrizing all overconvergent p-adic modular forms of fixed tame level and finite slope. Concurrently with their work, Stevens introduced his beautifully simple idea of *overconvergent cohomology* [Ste94], a group-cohomological avatar of overconvergent p-adic modular forms. Ash and Stevens developed these ideas much further in [AS08]: as conceived there, overconvergent cohomology works for any connected reductive $\mathbf{Q}$-group G split at p, and leads to natural candidates for quite general eigenvarieties. When the group $G^{\mathrm{der}}(\mathbf{R})$ possesses discrete series representations, Urban [Urb11] used overconvergent cohomology to construct eigenvarieties interpolating classical forms with nonzero Euler-Poincaré multiplicities, showing that his construction yields spaces which are equidimensional of the same dimension as weight space. In this article we develop new tools to analyze the situation for general groups.

To describe our results, we first introduce some notation. Fix a reductive $\mathbf{Q}$-group scheme G with $G^{\mathrm{der}}(\mathbf{R})$ noncompact, and fix a prime p such that G is split over $\mathbf{Q}_p$. Choose a Borel subgroup $B = TN$ and Iwahori subgroup I of $G(\mathbf{Q}_p)$. Let $\mathcal{W} = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbf{Z}_p), \mathbf{G}_m)^{\mathrm{an}}$ be the rigid analytic space of p-adically continuous characters of $T(\mathbf{Z}_p)$; as a rigid space, $\mathcal{W}$ is a disjoint union of open balls, each of dimension equal to the rank of G.[1] Multiplication of characters gives $\mathcal{W}$ the structure of a rigid analytic group. Given a continuous character $\lambda : T(\mathbf{Z}_p) \to \overline{\mathbf{Q}_p}^{\times}$, we also denote the corresponding point of $\mathcal{W}(\overline{\mathbf{Q}_p})$ by λ. A weight λ is *arithmetic* if $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ where λ^{alg} is a B-dominant algebraic character of T and ε is a character of finite order. In §2.1 we define, given an arbitrary affinoid open subset $\Omega \subset \mathcal{W}$ and an integer $s \gg 0$, an orthonormalizable Banach $A(\Omega)$-module $\mathbf{A}^s_\Omega$ and its dual module $\mathbf{D}^s_\Omega \simeq \mathrm{Hom}^{\mathrm{cts}}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Omega))$.[2] The module $\mathbf{A}^s_\Omega$ admits a canonical continuous left $A(\Omega)$-linear I-action, and $\mathbf{D}^s_\Omega$ inherits a corresponding right action. Furthermore, there are canonical $A(\Omega)[I]$-equivariant transition maps $\mathbf{A}^s_\Omega \to \mathbf{A}^{s+1}_\Omega$ which are injective and compact; we set

$$\mathcal{A}_\Omega = \lim_{s\to\infty} \mathbf{A}^s_\Omega$$

and

$$\mathcal{D}_\Omega = \lim_{\infty \leftarrow s} \mathbf{D}^s_\Omega.$$

For any rigid Zariski closed subspace $\Sigma \subset \Omega$, let $A(\Omega) \to A(\Sigma)$ denote the corresponding surjection on coordinate rings, and set $\mathcal{D}_\Sigma = \mathcal{D}_\Omega \otimes_{A(\Omega)} A(\Sigma)$. If λ is a classical dominant weight for B, with V_λ the corresponding irreducible right $G(\mathbf{Q}_p)$-representation of highest weight λ, there is a canonical continuous, surjective I-equivariant "integration" map $i_\lambda : \mathcal{D}_\Omega \to V_\lambda$ which factors through the map $\mathcal{D}_\Omega \to \mathcal{D}_\lambda$. There's no hope of finding a coherent sheaf of modules over Ω with the V_λ's as fibers, since the dimensions of the latter spaces vary unboundedly; in this light, $\mathcal{D}_\Omega$ p-adically interpolates the V_λ's in about the most straightforward way possible.

[1] By "rank" we shall always mean "absolute rank", i.e. the dimension of any maximal torus, split or otherwise.

[2] If X is an affinoid variety, $A(X)$ denotes the underlying affinoid algebra.

Let K_∞ be a maximal compact subgroup of $G(\mathbf{R})$, and let Z_∞ be the real points of the center of G. For any open compact subgroup $K^p \subset G(\mathbf{A}_f^p)$ in the prime-to-p finite adeles of G, the modules $\mathcal{A}_\Omega$ and $\mathcal{D}_\Omega$ give rise to local systems on the Shimura manifold $Y(K^pI) = G(\mathbf{Q})\backslash G(\mathbf{A})/K^pIK_\infty Z_\infty$. If M is any I-module, we write $H_*(K^p, M)$ and $H^*(K^p, M)$ for the homology and cohomology of the local system induced by M on $Y(K^pI)$. Ash and Stevens showed that the maps $\mathcal{D}_\lambda \to V_\lambda$ give rise to degree-preserving Hecke-equivariant morphisms

$$H^*(K^p, \mathcal{D}_\lambda) \to H^*(K^p, V_\lambda)$$

with cokernel contained in the space of forms of "critical slope". The target, by Matsushima's formula and its generalizations, is isomorphic as a Hecke module to a finite-dimensional space of classical automorphic forms; the source, on the other hand, is much larger. In the case of $G = \mathrm{GL}_2/\mathbf{Q}$, fundamental unpublished work of Stevens asserts that the modules $H^1(K^p, \mathcal{D}_\lambda)$ contain exactly the same finite-slope Hecke data as spaces of p-adic overconvergent modular forms, and that $H^1(K^p, \mathcal{D}_\Omega)$ interpolates the $H^1(K^p, \mathcal{D}_\lambda)$'s in a natural way. In general, the overconvergent cohomology modules $H^*(K^p, \mathcal{D}_\lambda)$ and their interpolations $H^*(K^p, \mathcal{D}_\Omega)$ seem to be an excellent surrogate for spaces of overconvergent p-adic modular forms.

Our first main result is an analogue of the universal coefficients theorem for overconvergent cohomology. As the duality functor $\mathrm{Hom}_{A(\Omega)}(-, A(\Omega))$ is far from exact, this result naturally takes the form of a spectral sequence. Before stating it, though, we need to explain one key difficulty. The structure of the individual $A(\Omega)$-modules $H_i(K^p, \mathcal{A}_\Omega)$ and $H^i(K^p, \mathcal{D}_\Omega)$ is very mysterious; in general they are not flat or finitely generated over $A(\Omega)$, and it seems unclear whether they're even Hausdorff in their natural topology. More precisely, $H_*(K^p, \mathcal{A}_\Omega)$ is the homology of a non-canonical chain complex $C_\bullet(K^p, \mathcal{A}_\Omega)$ of topological $A(\Omega)$-modules, and there's no particular reason for the boundaries of this complex to form a closed subspace of the cycles. To avoid this difficulty, as well the fact that the derived functors of $\mathrm{Hom}^{\mathrm{cts}}_{A(\Omega)}(-, A(\Omega))$ only make sense in the setting of *relative* homological algebra, we appeal to the theory of slope decompositions. More precisely, we define in §2.1 a monoid $\Delta \subset G(\mathbf{Q}_p)$ containing I which acts on the modules described above and extends their I-actions, and a submonoid $\Delta^+ \subset \Delta$ which acts completely continuously. The Hecke algebra $\mathcal{A}_p = \mathcal{H}_{\mathbf{Q}_p}(I\backslash\Delta/I)$ is commutative, and we define

$$\mathbf{T}(K^p) = \mathcal{A}_p \otimes_{\mathbf{Q}_p} \mathcal{H}_{\mathbf{Q}_p}(K^p\backslash G(\mathbf{A}_f^p)/K^p)^{\mathrm{sph}};$$

this is a commutative, non-Noetherian $\mathbf{Q}_p$-algebra, which acts on homology and cohomology in the usual way. For any element $t \in \Delta^+$, we set $U_t := [ItI] \in \mathbf{T}(K^p)$ and refer to U_t as a *controlling operator,* in analogy with the classical U_p-operator of Atkin and Lehner and its fundamental role in the theory of overconvergent modular forms. In §2.2 we explain how to lift the operator U_t from homology to a compact $A(\Omega)$-linear operator $\tilde{U}_t$ on the chain complex $C_\bullet(K^p, \mathcal{A}_\Omega)$. By combining some remarkable results of Buzzard and Ash-Stevens, we find that for any given rational number $h = a/b \in \mathbf{Q}_{\geq 0}$ and any point $x \in \mathcal{W}$, there is an affinoid $\Omega \subset \mathcal{W}$ containing x together with a *slope-$\leq h$ decomposition*

$$C_\bullet(K^p, \mathcal{A}_\Omega) \simeq C_\bullet(K^p, \mathcal{A}_\Omega)_{\leq h} \oplus C_\bullet(K^p, \mathcal{A}_\Omega)_{>h}$$

into closed $A(\Omega)[\tilde{U}_t]$-stable subcomplexes, where $C_\bullet(K^p, \mathcal{A}_\Omega)_{>h}$ is roughly the maximal $A(\Omega)$-subcomplex on which $p^{-a}\tilde{U}_t^b$ acts topologically nilpotently and the complex $C_\bullet(K^p, \mathcal{A}_\Omega)_{\leq h}$ is a

complex of finite flat $A(\Omega)$-modules (we give a precise, much more general definition of slope-$\leq h$ decompositions in §2.3). Passing to homology yields a decomposition of Hecke modules

$$H_*(K^p, \mathcal{A}_\Omega) \simeq H_*(K^p, \mathcal{A}_\Omega)_{\leq h} \oplus H_*(K^p, \mathcal{A}_\Omega)_{>h},$$

with $H_*(K^p, \mathcal{A}_\Omega)_{\leq h}$ finitely presented as an $A(\Omega)$-module. We define a *slope datum* as a triple (U_t, Ω, h) where U_t is a controlling operator, $\Omega \subset \mathcal{W}$ is a connected affinoid open subset, and $h \in \mathbf{Q}_{\geq 0}$ is such that $C_\bullet(K^p, \mathcal{A}_\Omega)$ admits a slope-$\leq h$ decomposition for the $\tilde{U}_t$-action. For the actual construction of eigenvarieties the choice of a controlling operator U_t is immaterial (and for many groups there is a canonical choice), so we shall sometimes implicitly fix a controlling operator and then refer to the pair (Ω, h) as a slope datum.

Theorem 1.1. *Fix a slope datum (U_t, Ω, h), and let $\Sigma \subseteq \Omega$ be an arbitrary rigid Zariski closed subspace. Then $H^*(K^p, \mathcal{D}_\Sigma)$ admits a slope-$\leq h$ decomposition, and there is a convergent first quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{A(\Omega)}(H_j(K^p, \mathcal{A}_\Omega)_{\leq h}, A(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\Sigma)_{\leq h}.$$

Furthermore, there is a convergent second quadrant spectral sequence

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{A(\Omega)}(H^j(K^p, \mathcal{D}_\Omega)_{\leq h}, A(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\Sigma)_{\leq h}.$$

In addition, there are analogous spectral sequences relating Borel-Moore homology with compactly supported cohomology, and boundary homology with boundary cohomology, and there are morphisms between the spectral sequences compatible with the morphisms between these different cohomology theories. Finally, the spectral sequences and the morphisms between them are equivariant for the natural Hecke actions on their E_2 pages and abutments; more succinctly, they are spectral sequences of $\mathbf{T}(K^p)$-modules.

When no ambiguity is likely, we will refer to the two spectral sequences of Theorem 1.1 as "the Ext spectral sequence" and "the Tor spectral sequence." We wish to emphasize that, unlike most familiar spectral sequences of universal coefficients type, these sequences have no particular reason to degenerate. However, the payoff of understanding their E_2 pages is very great: this knowledge is enough to calculate the dimensions of eigenvarieties. With the spectral sequences in hand, we prove the following basic result.

Theorem 1.2. *Fix a slope datum (U_t, Ω, h).*

i. For any i, $H_i(K^p, \mathcal{A}_\Omega)_{\leq h}$ is a faithful $A(\Omega)$-module if and only if $H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ is faithful.

ii. If G has $\mathbf{Q}$-anisotropic derived group, $H_i(K^p, \mathcal{A}_\Omega)_{\leq h}$ and $H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ are torsion $A(\Omega)$-modules for all i, unless $G^{\mathrm{der}}(\mathbf{R})$ has a discrete series, in which case they are torsion for all $i \neq \frac{1}{2}\dim G(\mathbf{R})/K_\infty Z_\infty$.

Some precursors of Theorem 1.2.ii were discovered by Hida (Theorem 5.2 of [Hid94], about which more below) and Ash-Pollack-Stevens [APS08]. We should note that in proving the non-faithfulness of $H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ in the discrete series case for $i \neq \frac{1}{2}\dim G(\mathbf{R})/K_\infty Z_\infty$, we make crucial use of the homology module $H_*(K^p, \mathcal{A}_\Omega)_{\leq h}$ and the Ext spectral sequence, and *we don't know how to prove this result by working solely with $\mathcal{D}_\Omega$*. This is a basic example of the benefit of treating the modules $\mathcal{A}_\Omega$ and $\mathcal{D}_\Omega$ on an equal footing.

Before turning to explain our motivations, let us say a word about the proof of the Ext spectral sequence of Theorem 1.1, the Tor spectral sequence being very similar. Given a commutative ring

R, an ideal $\mathfrak{a} \subset R$, an R-module M, and a chain complex $C_\bullet$ of projective R-modules, there is a quite general convergent spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R(H_j(C_\bullet), R/\mathfrak{a}) \Rightarrow H^{i+j}(\mathrm{Hom}_R(C_\bullet, R/\mathfrak{a})).$$

This result, however, isn't directly applicable towards proving Theorem 1.1 because we don't know whether or not $\mathbf{A}^s_\Omega$ is a projective $A(\Omega)$-module.[3] Fortunately, the theory of slope decompositions implies that $C_\bullet(K^p, \mathcal{A}_\Omega)_{\leq h}$ is a complex of finite flat and hence projective $A(\Omega)$-modules, from which Theorem 1.1 follows easily *except* for the assertion regarding Hecke equivariance. This latter property is absolutely crucial for applications, since it allows one to localize the entire spectral sequence at any ideal in the Hecke algebra. We proceed by constructing a homotopy action of $\mathbf{T}(K^p)$ directly on the complex $C_\bullet(K^p, \mathcal{A}_\Omega)_{\leq h}$ and making use of a derived-categorical point of view. Granted the preliminary materials of §2, the proof of Theorem 1.1 is actually rather short, and we refer the reader to §3 for details.

1.2 The geometry of eigenvarieties

Our main motivation for Theorem 1.1 is the problem, broadly speaking, of analyzing the geometry of eigenvarieties. To describe the spaces we have in mind, fix a controlling operator U_t. For any slope datum (Ω, h) we define $\mathbf{T}_{\Omega,h}(K^p)$ as the finite commutative $A(\Omega)$-algebra generated by the image of $\mathbf{T}(K^p) \otimes_{\mathbf{Q}_p} A(\Omega)$ in $\mathrm{End}_{A(\Omega)}\left(H^*(K^p, \mathcal{D}_\Omega)_{\leq h}\right)$. Given a Noetherian $\mathbf{Q}_p$-algebra A, an A-module M, and a $\mathbf{Q}_p$-algebra homomorphism $\phi : A \to \overline{\mathbf{Q}_p}$, we define the ϕ-generalized eigenspace $M_{(\phi)}$ of M as

$$M_{(\phi)} = \lim_{n \to \infty} M[\mathfrak{M}^n]$$

where $\mathfrak{M} = \ker \phi$.

Theorem 1.3. *There is a separated $\mathbf{Q}_p$-rigid analytic space $\mathscr{X} = \mathscr{X}(K^p)$ equipped with a (not necessarily surjective) morphism $w : \mathscr{X} \to \mathcal{W}$, locally finite in the domain over its image, with the following properties:*

i. There is a canonical algebra homomorphism $\psi : \mathbf{T}(K^p) \to \mathcal{O}(\mathscr{X})$, and the image of any double coset operator $[K^p g K^p]$ lies in the subring $\mathcal{O}(\mathscr{X})^{\leq 1}$ of power-bounded functions.

ii. Given any $\mathbf{Q}_p$-algebra homomorphism $\phi : \mathbf{T}(K^p) \to \overline{\mathbf{Q}_p}$, the ϕ-generalized eigenspace of $H^*(K^p, \mathcal{D}_\lambda)_{<\infty}$ is nonzero if and only if there is a point $x \in \mathscr{X}$ with $w(x) = \lambda$ such that the diagram

$$\begin{tikzcd} \mathbf{T}(K^p) \arrow[rr, "\phi"] \arrow[dr, "\psi(x)"'] & & \overline{\mathbf{Q}_p} \\ & \kappa(x) \arrow[ur, "\iota"'] & \end{tikzcd}$$

commutes for some embedding $\iota : \kappa(x) \to \overline{\mathbf{Q}_p}$. (Here $\kappa(x)$ is the residue field of $\mathcal{O}_{\mathscr{X},x}$, and $\psi(x)$ is the image of ψ under $\mathcal{O}(\mathscr{X}) \to \mathcal{O}_{\mathscr{X},x} \to \kappa(x)$.)

iii. The affinoid rigid spaces $\mathscr{X}_{\Omega,h} = \mathrm{Sp}\mathbf{T}_{\Omega,h}(K^p)$ admissibly cover $\mathscr{X}$ as (Ω, h) runs over all slope data.

iv. For each degree n, the $\mathbf{T}_{\Omega,h}(K^p)$-modules $H^n(K^p, \mathcal{D}_\Omega)_{\leq h}$ glue together over the aforementioned admissible covering of $\mathscr{X}(K^p)$ into a coherent $\mathcal{O}_{\mathscr{X}}$-module sheaf $\mathscr{M}^n$.

[3] This boils down to the question, seemingly open, of whether $\mathbf{Q}_p \langle X, Y \rangle$ is projective over $\mathbf{Q}_p \langle X \rangle$.

v. The space $\mathscr{X}$ is independent of the choice of controlling operator U_t used in its construction.

In other words, $\mathscr{X}$ analytically parametrizes the finite-slope eigenpackets occuring in $H^*(K^p, \mathcal{D}_\lambda)$ as λ varies over weight space. We sketch the construction of $\mathscr{X}$ in §3.2 - we stress, however, that *none* of the key ideas in this construction are due to us: they are due to Coleman-Mazur, Buzzard, and Ash-Stevens. We also note that for some groups, we can describe a canonical rigid subgroup $\mathcal{W}_0 \subset \mathcal{W}$, the *space of null weights*, together with a group-theoretic splitting $\mathcal{W} \to \mathcal{W}_0$ and a universal character $\mathbf{T}(K^p) \to \mathcal{O}(\mathcal{W}/\mathcal{W}_0)$ such that $\mathscr{X}$ is a $\mathcal{W}/\mathcal{W}_0$-bundle over $\mathscr{X}_0 = \mathscr{X} \cap w^{-1}(\mathcal{W}_0)$. Restricting to the space $\mathscr{X}_0$ amounts to factoring out wild twists, and is analogous to the Sen null space in Galois deformation theory. When $G = \mathrm{GL}_n/\mathbf{Q}$ and $T = \mathrm{diag}(t_1, \ldots, t_n)$ is the standard maximal torus, $\mathcal{W}_0$ is simply the space of characters trivial on the one-parameter subgroup $\mathrm{diag}(1, \ldots, 1, t_n)$. The first sign the construction described in Theorem 1.3 is worth consideration is probably the following theorem, which is due to Stevens [Ste00] and Bellaïche [Bel12].

Proposition 1.4. *When $G = \mathrm{GL}_2$ and $K^p = \mathrm{GL}_2(\mathbf{A}_f^p) \cap K_1(N)$, $\mathscr{X}_0$ is isomorphic to the Coleman-Mazur-Buzzard eigencurve $\mathscr{C}(N)$ of tame level N.*

In general, when $G^{\mathrm{der}}(\mathbf{R})$ has a discrete series, standard limit multiplicity results yield an abundance of classical automorphic forms of essentially every arithmetic weight, and one expects that correspondingly every irreducible component $\mathscr{X}_i$ of the eigenvariety $\mathscr{X}$ which contains a "suitably general" classical point has maximal dimension, namely $\dim \mathscr{X}_i = \dim \mathcal{W} = \mathrm{rank} G$. This numerical coincidence is *characteristic* of the groups for which $G^{\mathrm{der}}(\mathbf{R})$ has a discrete series. More precisely, define the *defect* and the *amplitude* of G, respectively, as the integers $l(G) = \mathrm{rank} G - \mathrm{rank} K_\infty Z_\infty$ and $q(G) = \frac{1}{2}(\dim(G(\mathbf{R})/K_\infty Z_\infty) - l(G))$. Note that $l(G)$ is zero if and only if $G^{\mathrm{der}}(\mathbf{R})$ has a discrete series, and that algebraic representations with regular highest weight contribute to $(\mathfrak{g}, K_\infty)$-cohomology exactly in the unbroken range of degrees $[q(G), q(G)+l(G)]$. We say a point $x \in \mathscr{X}(K^p)$ is *classical* if the induced eigenpacket $\psi(x) : \mathbf{T}(K^p) \to \kappa(x)$ matches the Hecke data of an algebraic automorphic representation π of $G(\mathbf{A_Q})$, and x is *regular* if π_∞ contributes to $(\mathfrak{g}, K_\infty)$-cohomology with coefficients in an irreducible algebraic representation with regular highest weight. The definition of a *non-critical* classical point is rather more subtle and we defer it until §2.4. The following conjecture is a special case of a conjecture of Urban (Conjecture 5.7.3 of [Urb11]).

Conjecture 1.5. *Every irreducible component $\mathscr{X}_i$ of $\mathscr{X}(K^p)$ containing a cuspidal non-critical regular classical point has dimension $\dim \mathcal{W} - l(G)$.*

To put this conjecture in context, we note that the inequality $\dim \mathscr{X}_i \geq \dim \mathcal{W} - l(G)$ is known. The idea for such a result is independently due to Stevens and Urban, whose proofs remain unpublished. On reading an earlier version of this paper, James Newton discovered a very short and elegant proof of this inequality, which is given here in Appendix B.

In any case, using the spectral sequences, we verify Conjecture 1.5 in many cases.

Theorem 1.6.

i. *If $l(G) = 0$, then Conjecture 1.5 is true, and if $x \in \mathscr{X}(K^p)$ is a cuspidal non-critical regular classical point, then $\mathscr{X}$ is smooth at x and the weight morphism w is étale at x.*

ii. *If $l(G) = 1$, then Conjecture 1.5 is true. Furthermore, if $l(G) \geq 1$, then every irreducible component of $\mathscr{X}(K^p)$ containing a cuspidal non-critical regular classical point has dimension at most $\dim \mathcal{W} - 1$.*

Some basic examples of groups with $l(G) = 1$ include $\mathrm{GL}_3/\mathbf{Q}$ and $\mathrm{Res}_{F/\mathbf{Q}} H$ where F is a number field with exactly one complex embedding and H is an F-inner form of GL_2/F. In particular, we have the following corollary, which was our original motivation for this project.

Corollary 1.7. *Let F be an imaginary quadratic extension of $\mathbf{Q}$, and let $G = \mathrm{Res}_{F/\mathbf{Q}}H$ where H is an F-inner form of GL_2/F (possibly the split form). Then there is a two-dimensional space of null weights $\mathscr{W}_0$, and any component of the eigenvariety $\mathscr{X}_0$ containing a cuspidal non-critical regular classical point is a rigid analytic curve.*

In the ordinary case, Corollary 1.7 is a result of Hida (§5-6 of [Hid94]). In fact, we are able to give a much more precise description of the geometry of the eigenvariety in this case.

Theorem 1.8. *Maintaining the notation and assumptions of Corollary 1.7, the eigenvariety $\mathscr{X}_0$ is naturally a union of subspaces $\mathscr{X}_0^{\mathrm{punc}} \cup \mathscr{X}_0^{\mathrm{Eis}} \cup \mathscr{X}_0^{\mathrm{cusp}}$ where:*

i. $\mathscr{X}_0^{\mathrm{punc}}$ is zero-dimensional, supported over the trivial weight in $\mathscr{W}_0$. The eigenpackets carried by $\mathscr{X}_0^{\mathrm{punc}}$ are of the form $T_{\mathfrak{q}} \mapsto (1+\mathbf{N}\mathfrak{q})\eta(\mathfrak{q})$, where η is an everywhere-unramified Hecke character of F.

ii. $\mathscr{X}_0^{\mathrm{Eis}}$ is empty unless H is F-split, in which case $\mathscr{X}_0^{\mathrm{Eis}}$ is finite and flat over $\mathscr{W}_0$. The fiber of $\mathscr{X}_0^{\mathrm{Eis}}$ over $\lambda \in \mathscr{W}_0$ carries the eigenpackets $T_{\mathfrak{q}} \mapsto \mathbf{N}\mathfrak{q} \cdot \lambda(\mathfrak{q})\eta_1(\mathfrak{q}) + \eta_2(\mathfrak{q})$, where η_1 and η_2 are certain finite-order Hecke characters of F of conductors dividing $\mathfrak{n}$.

iii. $\mathscr{X}_0^{\mathrm{cusp}}$ is equidimensional of dimension one.

The proofs of Theorem 1.2 and Theorem 1.6 systematically play the two spectral sequences against each other, and are curiously intertwined with the details of the *construction* of $\mathscr{X}(K^p)$; in particular, we make crucial use of a non-canonical intermediate rigid analytic space $\mathscr{Z}$ which depends heavily on the chain complex $C_\bullet(K^p, -)$. We also appeal to a simple but powerful analytic continuation principle: if $\mathscr{X}$ is an equidimensional rigid analytic space and $\mathscr{F}$ is a coherent analytic sheaf over $\mathscr{X}$ such that $\mathrm{Supp}\mathscr{F}$ contains *some* admissible open subset, then $\mathrm{Supp}\mathscr{F}$ contains an entire irreducible component of $\mathscr{X}$.

1.3 The relation to other incarnations of p-adic automorphic forms

It seems pertinent for us to make some remarks on the relation between overconvergent cohomology and other constructions of p-adic automorphic forms and eigenvarieties.

Aside from overconvergent cohomology, the other main approach to a "cohomological" construction of eigenvarieties is Emerton's completed cohomology [Eme06, CE12]. We briefly review the main definitions. Fix a tame level K^p and an open compact subgroup $K_p \subset G(\mathbf{Q}_p)$, and choose a filtration $K_p \supset K_p^1 \supset K_p^2 \supset \cdots \supset K_p^i \supset \ldots$ of K_p by open normal subgroups such that $\cap_{i=1}^{\infty} K_p^i = \{1\}$. We make the definitions

$$\widetilde{H}_i = \widetilde{H}_i(K^p) = \lim_{\infty \leftarrow j} H_i(G(\mathbf{Q})\backslash G(\mathbf{A})/K_p^j K^p K_\infty Z_\infty, \mathbf{Z}_p)$$

and

$$\widetilde{H}^i = \widetilde{H}^i(K^p) = \lim_{\infty \leftarrow k} \lim_{j \to \infty} H^i(G(\mathbf{Q})\backslash G(\mathbf{A})/K_p^j K^p K_\infty Z_\infty, \mathbf{Z}/p^k).$$

We write $\widetilde{H}_i^{BM}$ and $\widetilde{H}_c^i$ for the obvious Borel-Moore and compactly supported variants. The natural K_p-action on $\widetilde{H}^i$ and $\widetilde{H}_i$ extends to a continuous $G(\mathbf{Q}_p)$-action which is *independent* of the choice of K_p (explaining our omission of K_p from the notation). For each n, there are short exact sequences of continuous $G(\mathbf{Q}_p)$-modules

$$0 \to \mathrm{Hom}^{\mathrm{cts}}(\widetilde{H}_{n-1}, \mathbf{Q}_p/\mathbf{Z}_p) \to \widetilde{H}^n \to \mathrm{Hom}^{\mathrm{cts}}(\widetilde{H}_n, \mathbf{Z}_p) \to 0$$

and

$$0 \to \mathrm{Hom}(\widetilde{H}^{n+1}[p^\infty], \mathbf{Q}_p/\mathbf{Z}_p) \to \widetilde{H}_n \to \mathrm{Hom}^{\mathrm{cts}}(\widetilde{H}^n, \mathbf{Z}_p) \to 0.$$

In addition, for any fixed choice of K_p, the K_p-action on $\widetilde{H}_i$ extends to a left action of the (Noetherian) completed group ring $\Lambda = \mathbf{Z}_p[[K_p]]$ which gives $\widetilde{H}_i$ the structure of a finitely presented Λ-module. Finally, there is a spectral sequence of the form

$$E_2^{i,j} = \mathrm{Ext}^i_\Lambda(\widetilde{H}_j, \Lambda) \Rightarrow \widetilde{H}^{BM}_{d-i-j}$$

where $d = \mathrm{dim} G(\mathbf{R})/K_\infty Z_\infty$.

The following theorem and conjecture are due to Calegari and Emerton.

Theorem 1.9. *Suppose G is semisimple. Then $\widetilde{H}_i$ is a torsion Λ-module unless $G(\mathbf{R})$ has a discrete series and $i = \frac{1}{2}\mathrm{dim} G(\mathbf{R})/K_\infty Z_\infty$, in which case $\widetilde{H}_i$ is a faithful Λ-module.*

Conjecture 1.10. *Define the* codimension *of a Λ-module M, written* $\mathrm{codim}(M)$ *as the least integer i such that $\mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0$. Then $\mathrm{codim}(\widetilde{H}_{q(G)}) = l(G)$ and $\mathrm{codim}(\tilde{H}_i) > l(G)$ for $i \neq q(G)$.*

The reader will not fail to notice the very strong formal similarities between Theorem 1.2 and Theorem 1.9, and between Conjecture 1.5 and Conjecture 1.10. It seems very likely, to borrow a phrase from [CE12], that this relation “is more than one of mere analogy.” Some evidence for this has begun to accumulate: in a forthcoming paper [Han12a] we will construct Hecke-equivariant morphisms

$$\phi^n : H^n(K^p, \mathcal{A}_\lambda)_{<\infty} \to J_B^{T(\mathbf{Z}_p)=\lambda}(\widetilde{H}^n(K^p)_{\mathrm{la}}),$$

where $\lambda \in \mathcal{W}(\overline{\mathbf{Q}_p})$ is arbitrary, $\mathcal{A}_\lambda = \mathcal{A}_\Omega \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda$, $J_B^{T(\mathbf{Z}_p)=\lambda}(-)$ denotes the λ-isotypic subspace of Emerton’s locally analytic Jacquet module, and $(-)_{\mathrm{la}}$ denotes the functor of passage to $\mathbf{Q}_p$-locally analytic vectors defined in [ST03]. In many cases the morphisms ϕ^n are injective. Furthermore, the ϕ^n’s are defined as the edge maps of a spectral sequence - very different from those considered in this paper - which computes $H^*(K^p, \mathcal{A}_\lambda)$ using the continuous cohomology of the modules $\widetilde{H}^*(K^p)_{\mathrm{la}}$.

On the other hand, the completed cohomology groups should contain much more data than overconvergent cohomology: completed cohomology is expected to pick up essentially every Galois representation, while overconvergent cohomology should only pick up the trianguline Galois representations. Thus it is not obvious, at least to this author, how direct a connection between Conjectures 1.5 and 1.10 might be expected. In any case, the two theories seem to play complementary roles: completed cohomology is amenable to the powerful methods of p-adic analytic representation theory, while overconvergent cohomology with its theory of slope decompositions is planted firmly in the world of finitely generated modules over Noetherian rings.

Finally, despite the title of this paper, the reader may have noticed a conspicuous absence of overconvergent modular forms in our discussion. Let us simply say that when G gives rise to Shimura varieties, we expect that the Hecke data occuring in spaces of overconvergent modular forms for G will also occur in overconvergent cohomology, and that Chenevier’s interpolation theorem (Theorémè 1 of [Che05]) will allow a fairly straightforward verification of this expectation whenever a “small slope forms are classical”-type theorem is available.

Notation and terminology

Our notation and terminology is mostly standard. In nonarchimedian functional analysis and rigid analytic geometry we essentially follow [BGR84]. In the body of the paper, k denotes an extension field of $\mathbf{Q}_p$, complete for its norm $|\bullet|_k$. If M and N are k-Banach spaces, we write $\mathcal{L}_k(M, N)$ for the

space of continuous k-linear maps between M and N; the operator norm

$$|f| = \sup_{m \in M,\, |m|_M \leq 1} |f(m)|_N$$

makes $\mathcal{L}_k(M,N)$ into a k-Banach space. If $(A, |\bullet|_A)$ is a k-Banach space which furthermore is a commutative Noetherian k-algebra whose multiplication map is (jointly) continuous, we say A is a *k-Banach algebra*. An A-module M which is also a k-Banach space is a *Banach A-module* if the structure map $A \times M \to M$ extends to a continuous map $A \widehat{\otimes}_k M \to M$, or equivalently if the norm on M satisfies $|am|_M \leq C|a|_A|m|_M$ for all $a \in A$ and $m \in M$ with some fixed constant C. For a topological ring R and topological R-modules M, N, we write $\mathcal{L}_R(M,N)$ for the R-module of continuous R-linear maps $f : M \to N$. When A is a k-Banach algebra and M, N are Banach A-modules, we topologize $\mathcal{L}_A(M,N)$ via its natural Banach A-module structure. We write Ban_A for the category whose objects are Banach A-modules and whose morphisms are elements of $\mathcal{L}_A(-,-)$. If I is any set and A is a k-Banach algebra, we write $c_I(A)$ for the module of sequences $\mathbf{a} = (a_i)_{i \in I}$ with $|a_i|_A \to 0$; the norm $|\mathbf{a}| = \sup_{i \in I} |a_i|_A$ gives $c_I(A)$ the structure of a Banach A-module. If M is any Banach A-module, we say M is *orthonormalizable* if M is *isomorphic* to $c_I(A)$ for some I (such modules are called "potentially orthonormalizable" in [Buz07]).

If A is an affinoid algebra, then $\mathrm{Sp}A$, the *affinoid space* associated with A, denotes the locally G-ringed space $(\mathrm{Max}A, \mathcal{O}_A)$ where $\mathrm{Max}A$ is the set of maximal ideals of A endowed with the Tate topology and $\mathcal{O}_A$ is the extension of the assignment $U \mapsto A_U$, for affinoid subdomains $U \subset \mathrm{Max}A$ with representing algebras A_U, to a structure sheaf on $\mathrm{Max}A$. If X is an affinoid space, we write $A(X)$ for the coordinate ring of X. If A is reduced we equip A with the canonical supremum norm. If X is a rigid analytic space, we write $\mathcal{O}_X(X)$ or $\mathcal{O}(X)$ for the ring of global sections of the structure sheaf on X. Given a point $x \in X$, we write $\mathfrak{m}_x$ for the corresponding maximal ideal in $\mathcal{O}_X(U)$ for any admissible affinoid open $U \subset X$ containing x, and $\kappa(x)$ for the residue field $\mathcal{O}_X(U)/\mathfrak{m}_x$; $\mathcal{O}_{X,x}$ denotes the local ring of $\mathcal{O}_X$ at x in the Tate topology, and $\widehat{\mathcal{O}_{X,x}}$ denotes the $\mathfrak{m}_x$-adic completion of $\mathcal{O}_{X,x}$.

In homological algebra our conventions follow [Wei94]. If R is a ring, we write $\mathbf{K}^?(R)$, $? \in \{+,-,b,\emptyset\}$ for the homotopy category of ?-bounded R-module complexes and $\mathbf{D}^?(R)$ for its derived category.

Acknowledgments

This paper represents a portion of my forthcoming Ph.D. dissertation at Boston College. First and foremost, I'm grateful to my advisor, Avner Ash, for giving very generously of his time, insight, and enthusiasm. During the development of these ideas, I also enjoyed stimulating conversations with Joël Bellaïche, Jay Pottharst, Glenn Stevens, and Jack Thorne; it's a pleasure to acknowledge their help and influence here. I'm particularly grateful to Glenn Stevens for emphasizing the utility of slope decompositions to me, and for his kind encouragement. Kevin Buzzard, James Newton, and Jack Thorne offered helpful comments and corrections on earlier drafts, for which I thank them sincerely.

2 Background material

In §2.1-§2.3 we lay down some foundational notation and definitions; the ideas in these sections are entirely due to Ash and Stevens [AS08], though our presentation differs somewhat in insignificant details.

2.1 Algebraic groups and overconvergent coefficient modules

Lie theoretic data

Fix a prime p and a connected, reductive $\mathbf{Q}$-group G; we suppose $G/\mathbf{Q}_p$ is split and is the generic fiber of a smooth group scheme $\mathcal{G}/\mathbf{Z}_p$. Fix a Borel $B = TN$, an opposite Borel $B^{\mathrm{opp}} = TN^{\mathrm{opp}}$, and a compatible Iwahori subgroup $I \subset \mathcal{G}(\mathbf{Z}_p) \subset G(\mathbf{Q}_p)$. Set $X^* = \mathrm{Hom}(T, \mathbf{G}_m)$ and $X_* = \mathrm{Hom}(\mathbf{G}_m, T)$, and let Φ and Φ^+ be the sets of roots and positive roots, respectively, for the Borel B. We write X^*_+ for the cone of B-dominant weights; $\rho \in X^* \otimes_{\mathbf{Z}} \frac{1}{2}\mathbf{Z}$ denotes half the sum of the positive roots.

Set $N^\circ = \{n \in N^{\mathrm{opp}}(\mathbf{Z}_p),\, n \equiv 1 \,\mathrm{in}\, G(\mathbf{Z}/p\mathbf{Z})\}$, so the Iwahori decomposition reads $I = N^\circ \cdot T(\mathbf{Z}_p) \cdot N(\mathbf{Z}_p)$. For any integer $c \geq 1$, we set

$$I_0^c = \{g \in I,\, g \,\mathrm{mod}\, p^c \in B(\mathbf{Z}/p^c\mathbf{Z})\}$$

and

$$I_1^c = \{g \in I,\, g \,\mathrm{mod}\, p^c \in N(\mathbf{Z}/p^c\mathbf{Z})\}\,.$$

Note that I_1^c is normal in I_0^c, with quotient $T(\mathbf{Z}/p^c\mathbf{Z})$. For $s \geq 1$ a positive integer, define

$$I^s = \{g \in I | g \equiv 1 \,\mathrm{in}\, G(\mathbf{Z}/p^s\mathbf{Z})\}$$

and $T^s = T(\mathbf{Z}_p) \cap I^s$, $N^s = N(\mathbf{Z}_p) \cap I^s$. Note that N^s is normal in $B(\mathbf{Z}_p)$ and in $N(\mathbf{Z}_p)$.

We define semigroups of $T(\mathbf{Q}_p)$ by

$$\Lambda = \left\{t \in T(\mathbf{Q}_p),\, t^{-1}N(\mathbf{Z}_p)t \subseteq N(\mathbf{Z}_p)\right\}$$

and

$$\Lambda^+ = \left\{t \in T(\mathbf{Q}_p),\, \bigcap_{i=1}^{\infty} t^{-i}N(\mathbf{Z}_p)t^i = \{1\}\right\}.$$

A simple calculation shows that $t \in T(\mathbf{Q}_p)$ is contained in Λ (resp. Λ^+) if $v_p(\alpha(t)) \leq 0$ (resp. $v_p(\alpha(t)) < 0$) for all $\alpha \in \Phi^+$. Using these, we define semigroups of $G(\mathbf{Q}_p)$ by $\Delta = I\Lambda I$, $\Delta^+ = I\Lambda^+ I$. Note the inclusions $I \subset \Delta \supset \Delta^+$. The Iwahori decompsition extends to Δ: any element $g \in \Delta$ has a unique decomposition $g = \mathrm{n}^\circ(g)\mathrm{t}(g)\mathrm{n}(g)$ with $\mathrm{n}^\circ \in N^\circ$, $\mathrm{t} \in \Lambda$, $\mathrm{n} \in N(\mathbf{Z}_p)$. Fix once and for all a group homomorphism $\sigma : T(\mathbf{Q}_p) \to T(\mathbf{Z}_p)$ which splits the inclusion $T(\mathbf{Z}_p) \subset T(\mathbf{Q}_p)$. Note that σ splits Λ as the direct product $T(\mathbf{Z}_p) \cdot (\Lambda \cap \ker \sigma)$ via $\delta \mapsto (\sigma(\delta), \delta \cdot \sigma(\delta)^{-1})$.

We fix an analytic isomorphism $\psi : N(\mathbf{Z}_p) \simeq \mathbf{Z}_p^d$, $d = \mathrm{dim} N$, which identifies cosets of N^s with additive cosets $\mathbf{a} + p^s\mathbf{Z}_p^d \subset \mathbf{Z}_p^d$, $\mathbf{a} \in \mathbf{Z}_p^d$.

Definition. *If R is any $\mathbf{Q}_p$-Banach algebra and s is a nonnegative integer, the module $\mathbf{A}(N(\mathbf{Z}_p), R)^s$ of* s-locally analytic R-valued functions on $N(\mathbf{Z}_p)$ *is the R-module of continuous functions $f : N(\mathbf{Z}_p) \to R$ such that*

$$f_{\mathbf{a}} = f\left(\psi^{-1}\left(\mathbf{a} + p^s(x_1, \ldots, x_d)\right)\right) : \mathbf{Z}_p^d \to R$$

is given by an element of the d-variable Tate algebra $R\langle x_1, \ldots, x_d\rangle$ for any fixed $\mathbf{a} \in \mathbf{Z}_p^d$.

Given $f \in \mathbf{A}(N(\mathbf{Z}_p), R)^s$ and $\mathbf{i} = (i_1, \ldots, i_d) \in \mathbf{N}^d$, define coefficients $c(f_{\mathbf{a}}, \mathbf{i}) \in R$ by $f_{\mathbf{a}}(x_1, \ldots, x_d) = \sum_{\mathbf{i} \in \mathbb{N}^d} c(f_{\mathbf{a}}, \mathbf{i}) x_1^{i_1} \cdots x_d^{i_d}$. The norm

$$\|f\| = \sup_{\mathbf{a} \in \mathbf{Z}_p^d} \sup_{\mathbf{i} \in \mathbf{N}^d} |c(f_{\mathbf{a}}, \mathbf{i})|_R$$

defines a Banach R-module structure on $\mathbf{A}(N(\mathbf{Z}_p), R)^s$, with respect to which the canonical inclusion $\mathbf{A}(N(\mathbf{Z}_p), R)^s \subset \mathbf{A}(N(\mathbf{Z}_p), R)^{s+1}$ is compact.

Weights and Modules

The space of weights is the space $\mathcal{W} = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbf{Z}_p), \mathbf{G}_m)$; we briefly recall the rigid analytic structure on $\mathcal{W}$, following the discussion in §3.4-3.5 of [AS08]. For any rational number r, we define

$$\mathbf{Q}_p\langle p^r X\rangle = \left\{ \sum_{n=0}^{\infty} a_n X^n \in \mathbf{Q}_p[[X]],\ \lim_{n\to\infty} p^{rn}|a_n| \to 0 \right\};$$

this is a $\mathbf{Q}_p$-affinoid algebra, and $\mathcal{B}[p^r] = \mathrm{Sp}\mathbf{Q}_p\langle p^r X\rangle$ is the $\mathbf{Q}_p$-rigid analytic disk of radius p^r. Choose a monotone increasing sequence of rational numbers $\mathbf{r} = \{r_i\}_{i\in\mathbf{N}}$ with $r_i < 0$ and $\lim_{i\to\infty} r_i = 0$; the natural morphisms $\mathbf{Q}_p\langle p^{r_{i+1}} X\rangle \to \mathbf{Q}_p\langle p^{r_i} X\rangle$ dualize to maps $\mathcal{B}[p^{r_i}] \to \mathcal{B}[p^{r_{i+1}}]$, and we define *the $\mathbf{Q}_p$-rigid analytic open unit disk* $\mathcal{B}$ as the natural gluing of the $\mathcal{B}[p^{r_i}]$'s along these maps. The rigid structure on $\mathcal{B}$ is independent of the choice of rational sequence $\mathbf{r}$. There is a natural isomorphism $\mathcal{W} \simeq T\widehat{(\mathbf{Z}_p)_{\mathrm{tors}}} \times \mathcal{B}^d$, $d = \dim\mathbf{T}$, and we equip $\mathcal{W}$ with the unique rigid structure for which this is an isomorphism of rigid analytic spaces.

Let $\Omega \subset \mathcal{W}$ be an admissible open affinoid subset.

Lemma 2.1.1. *The ring $A(\Omega)$ is a regular ring.*

Proof. By construction, $\mathcal{W}$ is a disjoint union of finitely many open polydisks $\mathcal{B}^d$; write $\phi : \Omega \to \mathcal{B}^d \subset \mathcal{W}$ for the open immersion of Ω into whichever polydisk contains it. For any point $x \in \mathcal{B}^d$, the complete local ring $\widehat{\mathcal{O}_{\mathcal{B}^d,x}}$ is a power series ring in d variables over k_x, so is regular. For any $\omega \in \Omega$ the open immersion ϕ induces an isomorphism $\phi^* : \widehat{\mathcal{O}_{\mathcal{B}^d,\phi(\omega)}} \simeq \widehat{\mathcal{O}_{\Omega,\omega}}$, so the completed local rings of $A(\Omega)$ at all maximal ideals are regular. By Proposition 7.3.2/8 of [BGR84], this implies that each algebraic local ring $A(\Omega)_{\mathfrak{m}}$, $\mathfrak{m} \in \mathrm{Max}A(\Omega)$ is regular, as desired. □

We turn now to the key definitions of this section. Given $\Omega \subset \mathcal{W}$ admissible open, we write $\chi_\Omega : T(\mathbf{Z}_p) \to A(\Omega)^\times$ for the unique character it determines. We define $s[\Omega]$ as the minimal integer such that $\chi_\Omega|_{T^{s[\Omega]}}$ is analytic. For any integer $s \geq s[\Omega]$, we make the definition

$$\mathbf{A}_\Omega^s = \{f : I \to A(\Omega),\ f \,\text{analytic on each}\, I^s-\text{coset},\ f(n^\circ t g) = \chi_\Omega(t) f(g)\,\forall n^\circ \in N^\circ,\, t \in T(\mathbf{Z}_p),\, g \in I\}.$$

By the Iwahori decomposition, restricting an element $f \in \mathbf{A}_\Omega^s$ to $N(\mathbf{Z}_p)$ induces an isomorphism

$$\begin{array}{ccc} \mathbf{A}_\Omega^s & \simeq & \mathbf{A}(N(\mathbf{Z}_p), A(\Omega))^s \\ f & \mapsto & f|_{N(\mathbf{Z}_p)}, \end{array}$$

and we regard $\mathbf{A}_\Omega^s$ as a Banach $A(\Omega)$-module via pulling back the Banach module structure on $\mathbf{A}(N(\mathbf{Z}_p), A(\Omega))^s$ under this isomorphism. Right translation gives $\mathbf{A}_\Omega^s$ the structure of a continuous left $A(\Omega)[I]$-module. More generally, the formula

$$(N^\circ b) \star \delta = N^\circ \sigma(\delta)\delta^{-1} b\delta,\ b \in B(\mathbf{Z}_p) \simeq N^\circ\backslash I \,\text{and}\, \delta \in \Lambda$$

yields a right action of Δ on $N^\circ\backslash I$ which extends the natural right translation action by I (cf. §2.5 of [AS08]) and induces a left Δ-action on $\mathbf{A}^s_\Omega$ which we denote by $\delta \star f$, $f \in \mathbf{A}^s_\Omega$. For any $\delta \in \Delta^+$, the image of the operator $\delta \star - \in \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, \mathbf{A}^s_\Omega)$ factors through the inclusion $\mathbf{A}^{s-1}_\Omega \hookrightarrow \mathbf{A}^s_\Omega$, and so defines a completely continuous operator on $\mathbf{A}^s_\Omega$. The Banach dual

$$\begin{aligned}\mathbf{D}^s_\Omega &= \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Omega)) \\ &\simeq \mathcal{L}_{A(\Omega)}(\mathbf{A}(N(\mathbf{Z}_p), k)^s \widehat{\otimes}_k A(\Omega), A(\Omega)) \\ &\simeq \mathcal{L}_k(\mathbf{A}(N(\mathbf{Z}_p), k)^s, A(\Omega))\end{aligned}$$

inherts a dual right action of Δ, and the operator $\delta \star -$ for $\delta \in \Delta^+$ likewise factors through the inclusion $\mathbf{D}^{s+1}_\Omega \hookrightarrow \mathbf{D}^s_\Omega$.

We define

$$\mathcal{A}_\Omega = \lim_{s \to \infty} \mathbf{A}^s_\Omega$$

where the direct limit is taken with respect to the natural compact, injective transition maps $\mathbf{A}^s_\Omega \to \mathbf{A}^{s+1}_\Omega$. Note that $\mathcal{A}_\Omega$ is topologically isomorphic to the module of $A(\Omega)$-valued locally analytic functions on $N(\mathbf{Z}_p)$, equipped with the finest locally convex topology for which the natural maps $\mathbf{A}^s_\Omega \hookrightarrow \mathcal{A}_\Omega$ are continuous. The Δ-actions on $\mathbf{A}^s_\Omega$ induce a continuous Δ-action on $\mathcal{A}_\Omega$. Set

$$\mathcal{D}_\Omega = \{\mu : \mathcal{A}_\Omega \to A(\Omega),\, \mu\, \text{is}\, A(\Omega) - \text{linear and continuous}\},$$

and topologize $\mathcal{D}_\Omega$ via the coarsest locally convex topology for which the natural maps $\mathcal{D}_\Omega \to \mathbf{D}^s_\Omega$ are continuous. In particular, the canonical map

$$\mathcal{D}_\Omega \to \lim_{\infty \leftarrow s} \mathbf{D}^s_\Omega$$

is a topological isomorphism of locally convex $A(\Omega)$-modules, and $\mathcal{D}_\Omega$ is compact and Fréchet. Note that the transition maps $\mathbf{D}^{s+1}_\Omega \to \mathbf{D}^s_\Omega$ are *injective*, so $\mathcal{D}_\Omega = \cap_{s \gg 0} \mathbf{D}^s_\Omega$.

Suppose $\Sigma \subset \Omega$ is a Zariski closed subspace; by Corollary 9.5.2/8 of [BGR84], Σ arises from a surjection $A(\Omega) \twoheadrightarrow A(\Sigma)$ with $A(\Sigma)$ an affinoid algebra. We make the definitions $\mathbf{D}^s_\Sigma = \mathbf{D}^s_\Omega \otimes_{A(\Omega)} A(\Sigma)$ and $\mathcal{D}_\Sigma = \mathcal{D}_\Omega \otimes_{A(\Omega)} A(\Sigma)$.

Proposition 2.1.2. *There are canonical topological isomorphisms* $\mathbf{D}^s_\Sigma \simeq \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Sigma))$ *and* $\mathcal{D}_\Sigma \simeq \mathcal{L}_{A(\Omega)}(\mathcal{A}_\Omega, A(\Sigma))$.

Proof. Set $\mathfrak{a}_\Sigma = \ker(A(\Omega) \to A(\Sigma))$, so $A(\Sigma) \simeq A(\Omega)/\mathfrak{a}_\Sigma$. The definitions immediately imply isomorphisms

$$\begin{aligned}\mathbf{D}^s_\Sigma &\simeq \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Omega))/\mathfrak{a}_\Sigma \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Omega)) \\ &\simeq \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Omega))/\mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, \mathfrak{a}_\Sigma),\end{aligned}$$

so the first isomorphism will follow if we can verify that the sequence

$$0 \to \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, \mathfrak{a}_\Sigma) \to \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Omega)) \to \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Sigma))$$

is exact on the right. Given a k-Banach space E, write $b(E)$ for the Banach space of bounded sequences $\{(e_i)_{i \in \mathbb{N}}, \sup_{i \in \mathbb{N}} |e_i|_E < \infty\}$. Choosing an orthonormal basis of $\mathbf{A}(N(\mathbf{Z}_p), A(\Omega))^s$ gives rise to an isometry $\mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, E) \simeq b(E)$ for E any Banach $A(\Omega)$-module. Thus we need to show

the surjectivity of the reduction map $b(A(\Omega)) \to b(A(\Sigma))$. Choose a presentation $A(\Omega) = T_n/\mathfrak{b}_\Omega$, so $A(\Sigma) = T_n/\mathfrak{b}_\Sigma$ with $\mathfrak{b}_\Omega \subseteq \mathfrak{b}_\Sigma$. Quite generally for any $\mathfrak{b} \subset T_n$, the function

$$f \in T_n/\mathfrak{b} \mapsto \|f\|_{\mathfrak{b}} = \inf_{\tilde{f} \in f+\mathfrak{b}} \left\|\tilde{f}\right\|_{T_n}$$

defines a norm on $T_n/\mathfrak{b}$. By Proposition 3.7.5/3 of [BGR84], there is a unique Banach algebra structure on any affinoid algebra. Hence for any sequence $(f_i)_{i\in\mathbb{N}} \in b(A(\Sigma))$, we may choose a bounded sequence of lifts $(\widetilde{f}_i)_{i\in\mathbb{N}} \in b(T_n)$; reducing the latter sequence modulo $\mathfrak{b}_\Omega$, we are done.

Taking inverse limits in the sequence we just proved to be exact, the second isomorphism follows. □

For any complete extension $k/\mathbf{Q}_p$, let $\mathscr{A}_G(k)$ denote the ring of k-valued algebraic functions on $G(\mathbf{Q}_p)$; we write $\mathscr{A}_G = \mathscr{A}_G(\mathbf{Q}_p)$. We regard $\mathscr{A}_G(k)$ as a right $G(\mathbf{Q}_p)$-module via left translation of functions. Suppose $\lambda \in X^*_+ \subset \mathcal{W}(\mathbf{Q}_p)$ is a dominant weight for $\mathbf{B}$, with V_λ the corresponding irreducible right $G(\mathbf{Q}_p)$-representation of highest weight λ. The function v_λ defined on the big cell of the Bruhat decomposition by

$$v_\lambda(n'tn) = \lambda(t),\ (n', t, n) \in N^{opp}(\mathbf{Q}_p) \times T(\mathbf{Q}_p) \times N(\mathbf{Q}_p)$$

extends to a well-defined algebraic function $v_\lambda \in \mathscr{A}_G$. By the Borel-Weil-Bott theorem, the $\mathbf{Q}_p[N(\mathbf{Q}_p)]$-orbit of v_λ spans a canonical copy of V_λ inside $\mathscr{A}_G$ with v_λ a highest weight vector.

More generally, suppose $\lambda \in \mathcal{W}(k)$ is an arithmetic weight, with $\lambda = \lambda^{\mathrm{alg}} \cdot \epsilon$. Let $s = s[\lambda]$ be the smallest integer such that $T^s \subseteq \ker \epsilon$. Let $\Lambda^s \subset \Lambda$ be the monoid generated by $\Lambda \cap \ker \sigma$ and T^s, and set $\Delta^s = I_1^s \Lambda^s I_1^s$. We define f_λ by

$$f_\lambda(n'tn) = \lambda^{\mathrm{alg}}(t)\epsilon(\sigma(t)),\ (n', t, n) \in N^{opp}(\mathbf{Q}_p) \times T(\mathbf{Q}_p) \times N(\mathbf{Q}_p).$$

Note that $f_\lambda|_{\Delta^{s[\lambda]}} = v_{\lambda^{\mathrm{alg}}}|_{\Delta^{s[\lambda]}}$. The function $h \mapsto f_\lambda(hg)$, $h \in I_1^{s[\lambda]}$ and $g \in \Delta^{s[\lambda]}$ defines an element of $\mathbf{A}^s_\lambda \otimes_k V_{\lambda^{\mathrm{alg}}}$ for any $s \geq s[\lambda]$.

Proposition 2.1.3. *The formula*

$$v \star t = \lambda^{\mathrm{alg}}(t^{-1}\sigma(t)) v \cdot t,\ t \in \Lambda \,\mathrm{and}\, v \in V_{\lambda^{\mathrm{alg}}} \subset \mathcal{A}_G$$

extends to a well-defined right action of Δ *on* $V_{\lambda^{\mathrm{alg}}}$*, and the map*

$$\begin{array}{rcl} i_\lambda : \mathbf{D}^s_\lambda & \to & V_{\lambda^{\mathrm{alg}}} \\ \mu & \mapsto & i_\lambda(\mu) = \displaystyle\int f_\lambda(ng)\mu(n) \end{array}$$

is equivariant for the $\star$*-action of* $\Delta^{s[\lambda]}$.

For any $\delta \in \Lambda^{s[\lambda]}$, we calculate

$$\begin{aligned} i_\lambda(\mu \star \delta) &= \int f_\lambda(ng)(\mu \star \delta)(n) \\ &= \int f_\lambda(\sigma(\delta)\delta^{-1} n \delta g)\mu(n) \\ &= \lambda^{\mathrm{alg}}(\delta^{-1}\sigma(\delta))\epsilon(\sigma(\delta^{-1}\sigma(\delta))) \int f_\lambda(n\delta g)\mu(n) \\ &= \lambda^{\mathrm{alg}}(\delta^{-1}\sigma(\delta)) \int f_\lambda(n\delta g)\mu(n) \\ &= i_\lambda(\mu) \star \delta \end{aligned}$$

If $\gamma \in I_1^{s[\lambda]}$, the $\star$-actions are simply the usual actions, and we have

$$\begin{aligned} i_\lambda(\mu \star \gamma) &= \int f_\lambda(hg)(\mu \star \gamma)(h) \\ &= \int f_\lambda(h\gamma g)\mu(h) \\ &= i_\lambda(\mu) \star \gamma. \end{aligned}$$

□

The case of $\mathrm{GL}_n/\mathbf{Q}_p$

We examine the case of GL_n in detail. We choose B and B^{opp} as the upper and lower triangular Borel subgroups, respectively, and we identify T with diagonal matrices. The splitting σ is canonically induced from the homomorphism

$$\begin{aligned} \mathbf{Q}_p^\times &\to \mathbf{Z}_p^\times \\ x &\mapsto p^{-v_p(x)}x. \end{aligned}$$

Since $T(\mathbf{Z}_p) \simeq (\mathbf{Z}_p^\times)^n$, we canonically identify a character $\lambda : T(\mathbf{Z}_p) \to R^\times$ with the n-tuple of characters $(\lambda_1, \ldots, \lambda_n)$ where

$$\begin{aligned} \lambda_i : \mathbf{Z}_p^\times &\to R^\times \\ x &\mapsto \lambda \circ \mathrm{diag}(\underbrace{1, \ldots, 1}_{i-1}, x, 1, \ldots, 1). \end{aligned}$$

Dominant weights are identified with characters $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i(x) = x^{k_i}$ for integers $k_1 \geq k_2 \geq \cdots \geq k_n$.

We want to explain how to "twist away" one dimension's worth of weights in a canonical fashion. For any $\Omega \subset \mathcal{W}$, a simple calculation shows that the $\star$-action of Δ on $\mathbf{A}_\Omega^s$ is given explicitly by the formula

$$(\delta \star f)(x) = \chi_\Omega(\sigma(\mathrm{t}(\delta))\mathrm{t}(\delta)^{-1}\mathrm{t}(x\delta))f(\mathrm{n}(x\delta)),\ \delta \in \Delta,\ x \in N(\mathbf{Z}_p),\ f \in \mathbf{A}(N(\mathbf{Z}_p), A(\Omega))^s.$$

Given $1 \leq i \leq n$, let $m_i(g)$ denote the determinant of the upper-left i-by-i block of $g \in \mathrm{GL}_n$. For any $g \in \Delta$, a pleasant calculation left to the reader shows that

$$\mathrm{t}(g) = \mathrm{diag}(m_1(g), m_1(g)^{-1}m_2(g), \ldots, m_i^{-1}(g)m_{i+1}(g), \ldots, m_{n-1}(g)^{-1}\mathrm{det}g).$$

In particular, writing $\lambda^0 = (\lambda_1\lambda_n^{-1}, \lambda_2\lambda_n^{-1}, \ldots, \lambda_{n-1}\lambda_n^{-1}, 1)$ yields a canonical isomorphism

$$\mathbf{A}_\lambda^s \simeq \mathbf{A}_{\lambda^0}^s \otimes \lambda_n(\det \cdot |\det|_p)$$

of Δ-modules. If $\Gamma \subset \mathrm{GL}_n(\mathbf{Q})$ is a congruence lattice which satisfies $\det \Gamma = 1$, this descends to an isomorphism of Hecke modules

$$H_*(\Gamma, \mathbf{A}_\lambda^s) \simeq H_*(\Gamma, \mathbf{A}_{\lambda^0}^s) \otimes \lambda_n(\det \cdot |\det|_p).$$

As such we define the *null space* $\mathcal{W}^0 \subset \mathcal{W}$ as the subspace of weights which are trivial on the one-parameter subgroup $\mathrm{diag}(1, \ldots, 1, x)$, with its induced rigid analytic structure. Restricting our attention to weights in the null space amounts to factoring out central twists by "wild" characters.

In the case of GL_2 we can be even more explicit. Here Δ is generated by the center of $G(\mathbf{Q}_p)$ and by the monoid

$$\Sigma_0(p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p),\ c \in p\mathbf{Z}_p,\ a \in \mathbf{Z}_p^\times,\ ad - bc \neq 0 \right\}.$$

Another simple calculation shows that the center of $G(\mathbf{Q}_p)$ acts on $\mathbf{A}_\lambda^s$ through the character $z \mapsto \lambda(\sigma(z))$, while the monoid $\Sigma_0(p)$ acts via

$$(g \cdot f)(x) = (\lambda_1 \lambda_2^{-1})(a + cx)\lambda_2(\det g |\det g|_p) f\left(\frac{b + dx}{a + cx}\right),\ f \in \mathbf{A}(N(\mathbf{Z}_p), k)^s,\ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in N(\mathbf{Z}_p),$$

exactly as in [Ste94].

Remarks. There are some subtle differences between the different modules we have defined. The assignment $\Omega \mapsto \mathcal{A}_\Omega$ describes a presheaf over $\mathcal{W}$, and the modules $\mathbf{A}_\Omega^s$ are orthonormalizable. On the other hand, the modules $\mathbf{D}_\Omega^s$ are *not* obviously orthonormalizable, and $\mathcal{D}_\Omega$ doesn't obviously form a presheaf. There are alternate modules of distributions available, namely $\tilde{\mathbf{D}}_\Omega^s = \mathcal{L}_k(\mathbf{A}(N(\mathbf{Z}_p), k)^s, k)\widehat{\otimes}_k A(\Omega)$ and $\tilde{\mathcal{D}}_\Omega = \lim_{\leftarrow s} \tilde{\mathcal{D}}_\Omega^s$, equipped with suitable actions such that there is a natural $A(\Omega)[\Delta]$-equivariant embedding $\tilde{\mathcal{D}}_\Omega \hookrightarrow \mathcal{D}_\Omega$. The modules $\tilde{\mathbf{D}}_\Omega^s$ are orthonormalizable, and $\tilde{\mathcal{D}}_\Omega$ forms a presheaf over weight space, but of course is not the continuous dual of $\mathcal{A}_\Omega$. Despite these differences, the practical choice to work with one module or the other is really a matter of taste: for any $\lambda \in \Omega(\overline{\mathbf{Q}_p})$, there are isomorphisms

$$\mathcal{D}_\Omega \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda \simeq \tilde{\mathcal{D}}_\Omega \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda \simeq \mathcal{D}_\lambda,$$

and in point of fact the slope-$\leq h$ subspaces of $H^*(K^p, \mathcal{D}_\Omega)$ and $H^*(K^p, \tilde{\mathcal{D}}_\Omega)$, when they are defined, are canonically isomorphic as Hecke modules, as we show in Proposition 2.4.6 below. One of our implicit goals in this paper is to demonstrate the feasibility of working successfully with the modules $\mathcal{D}_\Omega$ by treating the dual modules $\mathcal{A}_\Omega$ on an equal footing.

2.2 Shimura manifolds and cohomology of local systems

In this section we set up our conventions for the homology and cohomology of local systems on Shimura manifolds. Following [AS08], we compute homology and cohomology using two different families of resolutions: some extremely large "adelic" resolutions which have the advantage of making the Hecke action transparent, and resolutions with good finiteness properties constructed from simplicial decompositions of the Borel-Serre compactifications of locally symmetric spaces.

Resolutions and complexes

Let $G/\mathbf{Q}$ be a connected reductive group. Let $G(\mathbf{R})^\circ$ denote the connected component of $G(\mathbf{R})$ containing the identity element, with $G(\mathbf{Q})^\circ = G(\mathbf{Q}) \cap G(\mathbf{R})^\circ$. Fix a maximal compact subgroup $K_\infty \subset G(\mathbf{R})$ with K_∞° the connected component containing the identity, and let Z_∞ denote the real

points of a maximal $\mathbf{Q}$-split torus contained in the center of G. Given an open compact subgroup $K_f \subset G(\mathbf{A}_f)$, we define the *Shimura manifold of level* K_f by

$$Y(K_f) \quad = \quad G(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty^\circ Z_\infty.$$

This is a possibly disconnected Riemannian orbifold. By strong approximation there is a finite set of elements $\gamma(K_f) = \{x_i, x_i \in G(\mathbf{A}_f)\}$ with

$$G(\mathbf{A}) = \coprod_{x_i \in \gamma(K_f)} G(\mathbf{Q})^\circ G(\mathbf{R})^\circ x_i K_f.$$

Defining $\Gamma(x_i) = G(\mathbf{Q})^\circ \cap x_i K_f x_i^{-1}$, we have a decomposition

$$Y(K_f) = G(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty^\circ Z_\infty \simeq \coprod_{x_i \in \gamma(K_f)} \Gamma(x_i)\backslash D_\infty,$$

where $D_\infty = G(\mathbf{R})^\circ/K_\infty^\circ Z_\infty$ is the symmetric space associated with G. We shall assume for the remainder of this paper that K_f is *neat:* the image of each $\Gamma(x_i)$ in $G^{\mathrm{ad}}(\mathbf{R})$ is torsion-free, in which case $Y(K_f)$ is a smooth manifold. If N is any right K_f-module, the double quotient

$$\widetilde{N} = G(\mathbf{Q})\backslash \left(D_\infty \times G(\mathbf{A}_f) \times N\right)/K_f$$

naturally gives rise to a local system on $Y(K_f)$. Set $D_\mathbf{A} = D_\infty \times G(\mathbf{A}_f)$, and let $C_\bullet(D_\mathbf{A})$ denote the complex of singular chains on $D_\mathbf{A}$ endowed with the natural action of $G(\mathbf{Q}) \times G(\mathbf{A}_f)$. If M and N are left and right K_f-modules, respectively, we define the complexes of *adelic chains* and *adelic cochains* by

$$C_\bullet^{ad}(K_f, M) = C_\bullet(D_\mathbf{A}) \otimes_{\mathbf{Z}[G(\mathbf{Q})\times K_f]} M$$

and

$$C^\bullet_{ad}(K_f, N) = \mathrm{Hom}_{\mathbf{Z}[G(\mathbf{Q})\times K_f]}(C_\bullet(D_\mathbf{A}), N).$$

Proposition 2.2.1. *There is a canonical isomorphism*

$$H^*(Y(K_f), \widetilde{N}) = H^*(C^\bullet_{ad}(K_f, N)).$$

Proof. Let $C_\bullet(D_\infty)(x_i)$ denote the complex of singular chains on D_∞, endowed with the natural left action of $\Gamma(x_i)$ induced from the left action of $G(\mathbf{Q})$ on D_∞; since D_∞ is contractible, this is a free resolution of $\mathbf{Z}$ in the category of $\mathbf{Z}[\Gamma(x_i)]$-modules. Let $N(x_i)$ denote the left $\Gamma(x_i)$-module whose underlying module is N but with the action $\gamma \cdot_{x_i} n = n|x_i^{-1}\gamma^{-1}x_i$. Note that the local system $\widetilde{N}(x_i)$ obtained by restricting $\widetilde{N}$ to the connected component $\Gamma(x_i)\backslash D_\infty$ of $Y(K_f)$ is simply the quotient $\Gamma(x_i)\backslash \left(D_\infty \times N(x_i)\right)$. Setting

$$C^\bullet_{sing}(K_f, N) = \oplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma(x_i)]}(C_\bullet(D_\infty)(x_i), N(x_i)),$$

the map $D_\infty \to (D_\infty, x_i) \subset D_\mathbf{A}$ induces a morphism $x_i^* = \mathrm{Hom}(C_\bullet(D_\mathbf{A}), N) \to \mathrm{Hom}(C_\bullet(D_\infty), N)$, which in turn induces an isomorphism

$$\oplus_i x_i^* : C^\bullet_{ad}(K_f, N) \xrightarrow{\sim} \oplus_i \mathrm{Hom}_{\Gamma(x_i)}(C_\bullet(D_\infty)(x_i), N(x_i)),$$

and passing to cohomology we have

$$\begin{aligned} H^*(C^\bullet_{ad}(K_f,N)) &\simeq \oplus_i H^*(\Gamma(x_i)\backslash D_\infty, \widetilde{N}(x_i)) \\ &\simeq H^*(Y(K_f),\widetilde{N}) \end{aligned}$$

as desired. □

For each $x_i \in \gamma(K_f)$, we choose a finite resolution $F_\bullet(x_i) \to \mathbf{Z} \to 0$ of $\mathbf{Z}$ by free left $\mathbf{Z}[\Gamma(x_i)]$-modules of finite rank as well as a homotopy equivalence $F_\bullet(x_i) \underset{g_i}{\overset{f_i}{\rightleftarrows}} C_\bullet(D_\infty)(x_i)$. We shall refer to the resolution $F_\bullet(x_i)$ as a *Borel-Serre resolution*; the existence of such resolutions follows from taking a finite simplicial decomposition of the Borel-Serre compactification of $\Gamma(x_i)\backslash D_\infty$. Setting

$$C_\bullet(K_f,N) = \oplus_i F_\bullet(x_i) \otimes_{\mathbf{Z}[\Gamma(x_i)]} M(x_i)$$

and

$$C^\bullet(K_f,N) = \oplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma(x_i)]}(F_\bullet(x_i), N(x_i)),$$

the maps f_i, g_i induce homotopy equivalences

$$C_\bullet(K_f,M) \underset{g_*}{\overset{f_*}{\rightleftarrows}} C^{ad}_\bullet(K_f,M)$$

and

$$C^\bullet(K_f,N) \underset{f^*}{\overset{g^*}{\rightleftarrows}} C^\bullet_{ad}(K_f,M).$$

We refer to the complexes $C_\bullet(K_f,-)$ and $C^\bullet(K_f,-)$ as *Borel-Serre complexes*, and we refer to these complexes together with a *fixed* set of homotopy equivalences $\{f_i, g_i\}$ as *augmented Borel-Serre complexes.*

Hecke operators

Suppose now that R is a commutative ring, $\Delta \subset G(\mathbf{A}_f)$ is a monoid containing K_f, and M is a left $R[\Delta]$-module. The complex $C_\bullet(D_{\mathbf{A}}) \otimes_{\mathbf{Z}[G(\mathbf{Q})]} M$ receives a Δ-action via $\delta \cdot (\sigma \otimes m) = \sigma\delta^{-1} \otimes \delta m$, and $C^{ad}_\bullet(K_f,M)$ is naturally identified with the K_f-coinvariants of this action. Given any double coset $K_f\delta K_f = \coprod_j K_f\delta_j$, the action defined on pure tensors by the formula

$$[K_f\delta K_f] \cdot (\sigma \otimes m) = \sum_j \delta_j \cdot (\sigma \otimes m)$$

induces a well-defined algebra homomorphism

$$\xi : \mathcal{H}_R(K_f\backslash\Delta/K_f) \to \mathrm{End}_R(C^{ad}_\bullet(K_f,M)).$$

This action commutes with the boundary maps, and induces the usual Hecke action defined by correspondences on cohomology. The map

$$\begin{aligned} \tilde{\xi} : \mathcal{H}_R(K_f\backslash\Delta/K_f) &\to \mathrm{End}_{\mathbf{K}(R)}(C_\bullet(K_f,M)) \\ T &\mapsto g_* \circ \xi(T) \circ f_* \end{aligned}$$

is well-defined, since f_* and g_* are well-defined up to homotopy equivalence. For any Hecke operator T, we will abbreviate $\tilde{\xi}(T)$ by $\tilde{T}$. Note that any individual lift $\tilde{T}$ is well-defined in $\mathrm{End}_R(C_\bullet(K_f, M))$, but if T_1 and T_2 commute in the abstract Hecke algebra, $\tilde{T}_1\tilde{T}_2$ and $\tilde{T}_2\tilde{T}_1$ will typically only commute up to homotopy.

Likewise, if N is a right Δ-module, the complex $\mathrm{Hom}_{\mathbf{Z}[G(\mathbf{Q})]}(C_\bullet(D_\mathbf{A}), N)$ receives a natural Δ-action via the formula $\phi|\delta = \phi(\sigma\delta^{-1})\delta$, and $C^\bullet_{ad}(K_f, N)$ is naturally the K_f-invariants of this action. The formula

$$\phi|[K_f\delta K_f] = \sum_j \phi|\delta_j$$

yields an algebra homomorphism $\xi : \mathcal{H}_R(K_f\backslash\Delta/K_f) \to \mathrm{End}_R(C^\bullet_{ad}(K_f, N))$ which induces the usual Hecke action on cohomology, and $\tilde{\xi} = f^* \circ \xi \circ g^*$ defines an algebra homomorphism $\mathcal{H}_R(K_f\backslash\Delta/K_f) \to \mathrm{End}_{\mathbf{K}(R)}(C^\bullet(K_f, M))$.

It is extremely important for us that these Hecke actions are compatible with the duality isomorphism

$$\mathrm{Hom}_R(C_\bullet(K_f, M), P) \simeq C^\bullet(K_f, \mathrm{Hom}_R(M, P)),$$

where P is any R-module.

2.3 Slope decompositions of modules and complexes

Here we review the very general notion of slope decomposition introduced in [AS08]. Let A be a k-Banach algebra, and let M be an A-module equipped with an A-linear endomorphism $u : M \to M$ (for short, "an $A[u]$-module"). Fix a rational number $h \in \mathbf{Q}_{\geq 0}$. We say a polynomial $Q \in A[x]$ is *multiplicative* if the leading coefficient of Q is a unit in A, and that Q has *slope* $\leq h$ if every edge of the Newton polygon of Q has slope $\leq h$. Write $Q^*(x) = x^{\deg Q}Q(1/x)$. An element $m \in M$ has slope $\leq h$ if there is a multiplicative polynomial $Q \in A[T]$ of slope $\leq h$ such that $Q^*(u) \cdot m = 0$. Let $M_{\leq h}$ be the set of elements of M of slope $\leq h$; according to Proposition 4.6.2 of *loc. cit.*, $M_{\leq h}$ is an A-submodule of M.

Definition 2.3.1. A *slope-$\leq h$ decomposition* of M is an $A[u]$-module isomorphism

$$M \simeq M_{\leq h} \oplus M_{>h}$$

such that $M_{\leq h}$ is a finitely generated A-module and the map $Q^*(u) : M_{>h} \to M_{>h}$ is an A-module isomorphism for every multiplicative polynomial $Q \in A[T]$ of slope $\leq h$.

The following proposition summarizes the fundamental results on slope decompositions.

Proposition 2.3.1 (Ash-Stevens):

a) *Suppose M and N are both $A[u]$-modules with slope-$\leq h$ decompositions. If $\psi : M \to N$ is a morphism of $A[u]$-modules, then $\psi(M_{\leq h}) \subseteq N_{\leq h}$ and $\psi(M_{>h}) \subseteq N_{>h}$; in particular, a module can have at most one slope-$\leq h$ decomposition. Furthermore,* $\ker\psi$ *and* $\mathrm{im}\psi$ *inherit slope-$\leq h$ decompositions. Given a short exact sequence*

$$0 \to M \to N \to L \to 0$$

of $A[u]$-modules, if two of the modules admit slope-$\leq h$ decompositions then so does the third.

b) *If $C^\bullet$ is a complex of $A[u]$-modules, all with slope-$\leq h$ decompositions, then*

$$H^n(C^\bullet) \simeq H^n(C^\bullet_{\leq h}) \oplus H^n(C^\bullet_{>h})$$

is a slope-$\leq h$ decomposition of $H^n(C^\bullet)$.

Proof. This is a rephrasing of (a specific case of) Proposition 4.1.2 of *loc. cit.* □

Suppose now that A is a reduced affinoid algebra, M is an orthonormalizable Banach A-module, and u is a compact operator. Let

$$F(T) = \det(1 - uT)|M \in A[[T]]$$

denote the Fredholm determinant for the u-action on M. We say F admits a *slope-$\leq h$ factorization* if we can write $F(T) = Q(T) \cdot R(T)$ where Q is a multiplicative polynomial of slope $\leq h$ and $R(T) \in A[[T]]$ is an entire power series of slope $> h$. Theorem 3.3 of [Buz07] guarantees that F admits a slope-$\leq h$ factorization if and only if M admits a slope-$\leq h$ decomposition. Furthermore, given a slope-$\leq h$ factorization $F(T) = Q(T) \cdot R(T)$, we obtain the slope-$\leq h$ decomposition of M upon setting $M_{\leq h} = \{m \in M | Q^*(u) \cdot m = 0\}$, and $M_{\leq h}$ in this case is a finite flat A-module upon which u acts invertibly.[4] Combining this with Theorem 4.5.1 of [AS08] and Proposition 2.3.1, we deduce:

Proposition 2.3.2. *If $C^\bullet$ is a bounded complex of orthonormalizable Banach $A[u]$-modules, and u acts compactly on the total complex $\oplus_i C^i$, then for any $x \in \mathrm{Max}(A)$ and any $h \in \mathbf{Q}_{\geq 0}$ there is an affinoid subdomain $\mathrm{Max}(A') \subset \mathrm{Max}(A)$ containing x such that the complex $C^\bullet \widehat{\otimes}_A A'$ of $A'[u]$-modules admits a slope-$\leq h$ decomposition, and $(C^\bullet \widehat{\otimes}_A A')_{\leq h}$ is a complex of finite flat A'-modules.*

Proposition 2.3.3. *If M is an orthonormalizable Banach A-module with a slope-$\leq h$ decomposition, and A' is a Banach A-algebra, then $M \widehat{\otimes}_A A'$ admits a slope-$\leq h$ decomposition and in fact*

$$(M \widehat{\otimes}_A A')_{\leq h} \simeq M_{\leq h} \otimes_A A'.$$

Proposition 2.3.4. *If $N \in \mathrm{Ban}_A$ is finite and $M \in \mathrm{Ban}_A$ is an $A[u]$-module with a slope-$\leq h$ decomposition, the $A[u]$-modules $M \widehat{\otimes}_A N$ and $\mathcal{L}_A(M, N)$ inherit slope-$\leq h$ decompositions.*

Proof. This is an immediate consequence of the A-linearity of the u-action and the fact that $-\widehat{\otimes}_A N$ and $\mathcal{L}_A(-, N)$ commute with finite direct sums. □

2.4 Overconvergent (co)homology

In this section we establish some foundational results on overconvergent cohomology. These results likely follow from the formalism introduced in Chapter 5 of [AS08], but we give different proofs. We use the notations introduced in §2.1-§2.3.

Fix a connected, reductive group $G/\mathbf{Q}$ with $G/\mathbf{Q}_p$ split, and fix a tame level $K^p \subset G(\mathbf{A}_f^p)$; in the presence of overconvergent coefficient modules, we *always* take our wild level subgroup of $G(\mathbf{Q}_p)$ to be I, which we drop from the notation, writing e.g. $H_*(K^p, \mathcal{A}_\Omega) = H_*(K^p I, \mathcal{A}_\Omega)$. Fix an augmented Borel-Serre complex $C_\bullet(K^p, -) = C_\bullet(K^p I, -)$. Fix an element $t \in \Delta^+$, and let $\tilde{U} = \tilde{U}_t$ denote the lifting of $U_t = [ItI]$ to an endomorphism of the complex $C_\bullet(K^p, -)$ defined in §2.2. Given

[4] Writing $Q^*(x) = a + x \cdot r(x)$ with $r \in A[x]$ and $a \in A^\times$, u^{-1} on $M_{\leq h}$ is given explicitly by $-a^{-1} r(u)$.

a connected admissible open affinoid subset $\Omega \subset \mathcal{W}$ and any integer $s \geq s[\Omega]$, the endomorphism $\tilde{U}_t \in \mathrm{End}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega))$ is completely continuous; let

$$F^s_\Omega(T) = \det(1 - T\tilde{U}_t)|C_\bullet(K^p, \mathbf{A}^s_\Omega) \in A(\Omega)[[T]]$$

denote its Fredholm determinant. We say (U_t, Ω, h) is a *slope datum* if $C_\bullet(K^p, \mathbf{A}^s_\Omega)$ admits a slope-$\leq h$ decomposition for the $\tilde{U}_t$ action for some $s \geq s[\Omega]$; we shall see shortly that this agrees with the definition given in the introduction.

Proposition 2.4.1. *The function $F^s_\Omega(T)$ is independent of s.*

Proof. For any integer $s \geq s[\Omega]$ we write $C^s_\bullet = C_\bullet(K^p, \mathbf{A}^s_\Omega)$ for brevity. By construction, the operator $\tilde{U}_t$ factors into compositions $\rho_s \circ \check{U}_t$ and $\check{U}_t \circ \rho_{s+1}$ where $\check{U}_t : C^s_\bullet \to C^{s-1}_\bullet$ is continuous and $\rho_s : C^{s-1}_\bullet \to C^s_\bullet$ is completely continuous. Now, considering the commutative diagram[5]

$$\begin{tikzcd}
C^s_\bullet \arrow[r, "\check{U}_t"] \arrow[d, "\tilde{U}_t"'] & C^{s-1}_\bullet \arrow[d, "\check{U}_t"] \arrow[dl, "\rho_s"'] \\
C^s_\bullet \arrow[r, "\check{U}_t"'] & C^{s-1}_\bullet
\end{tikzcd}$$

we calculate

$$\begin{aligned}
\det(1 - T\tilde{U}_t)|C^s_\bullet &= \det(1 - T\rho_s \circ \check{U}_t)|C^s_\bullet \\
&= \det(1 - T\check{U}_t \circ \rho_s)|C^{s-1}_\bullet \\
&= \det(1 - T\tilde{U}_t)|C^{s-1}_\bullet,
\end{aligned}$$

where the second line follows from Lemma 2.7 of [Buz07], so $F^s_\Omega(T) = F^{s-1}_\Omega(T)$ for all $s > s[\Omega]$. □

Proposition 2.4.2. *The slope-$\leq h$ subcomplex $C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}$, if it exists, is independent of s. If Ω' is an affinoid subdomain of Ω, then there is a canonical isomorphism*

$$C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega') \simeq C_\bullet(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}$$

for any $s \geq s[\Omega]$.

Proof. Since $F^s_\Omega(T)$ is independent of s, we simply write $F_\Omega(T)$. Suppose we are given a slope-$\leq h$ factorization $F_\Omega(T) = Q(T) \cdot R(T)$; by the remarks in §2.3, setting $C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} = \ker Q^*(\tilde{U}_t)$ yields a slope-$\leq h$ decomposition of $C_\bullet(K^p, \mathbf{A}^s_\Omega)$ for any $s \geq s[\Omega]$. By Proposition 2.3.1, the injection $\rho_s : C_\bullet(K^p, \mathbf{A}^{s-1}_\Omega) \hookrightarrow C_\bullet(K^p, \mathbf{A}^s_\Omega)$ gives rise to a canonical injection

$$\rho_s : C_\bullet(K^p, \mathbf{A}^{s-1}_\Omega)_{\leq h} \hookrightarrow C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}$$

for any $s > s[\Omega]$. The operator $\tilde{U}_t$ acts invertibly on $C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}$, and its image factors through ρ_s, so ρ_s is surjective and hence bijective. This proves the first claim.

For the second claim, by Proposition 2.3.3 we have

$$\begin{aligned}
C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega') &\simeq \left(C_\bullet(K^p, \mathbf{A}^s_\Omega)\widehat{\otimes}_{A(\Omega)} A(\Omega')\right)_{\leq h} \\
&\simeq C_\bullet(K^p, \mathbf{A}^s_{\Omega'})_{\leq h},
\end{aligned}$$

[5] The importance of drawing diagrams like this seems to have first been realized by Hida.

so the result now follows from the first claim. □

Proposition 2.4.3. *Given a slope datum (U_t, Ω, h) and an affinoid subdomain $\Omega' \subset \Omega$, there is a canonical isomorphism*

$$H_*(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega') \simeq H_*(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}$$

for any $s \geq s[\Omega]$.

Proof. Since $A(\Omega')$ is $A(\Omega)$-flat, the functor $- \otimes_{A(\Omega)} A(\Omega')$ commutes with taking the homology of any complex of $A(\Omega)$-modules. Thus we calculate

$$\begin{aligned} H_*(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega) &\simeq H_*\left(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h}\right) \otimes_{A(\Omega)} A(\Omega') \\ &\simeq H_*\left(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega')\right) \\ &\simeq H_*\left(C_\bullet(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}\right) \\ &\simeq H_*(K^p, \mathbf{A}^s_{\Omega'})_{\leq h}, \end{aligned}$$

where the third line follows from Proposition 2.3.3. □

Proposition 2.4.4. *Given a slope datum (U_t, Ω, h), the complex $C_\bullet(K^p, \mathcal{A}_\Omega)$ and the homology module $H_*(K^p, \mathcal{A}_\Omega)$ admit slope-$\leq h$ decompositions, and there is an isomorphism*

$$H_*(K^p, \mathcal{A}_\Omega)_{\leq h} \simeq H_*(K^p, \mathbf{A}^s_\Omega)_{\leq h}$$

for any $s \geq s[\Omega]$. Furthermore, given an affinoid subdomain $\Omega' \subset \Omega$, there is a canonical isomorphism

$$H_*(K^p, \mathcal{A}_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega') \simeq H_*(K^p, \mathcal{A}_{\Omega'})_{\leq h}.$$

Proof. For any fixed $s \geq s[\Omega]$, we calculate

$$\begin{aligned} C_\bullet(K^p, \mathcal{A}_\Omega) &\simeq \varinjlim_{s'} C_\bullet(K^p, \mathbf{A}^{s'}_\Omega) \\ &\simeq \varinjlim_{s'} C_\bullet(K^p, \mathbf{A}^{s'}_\Omega)_{\leq h} \oplus C_\bullet(K^p, \mathbf{A}^{s'}_\Omega)_{>h} \\ &\simeq C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\leq h} \oplus \varinjlim_{s'} C_\bullet(K^p, \mathbf{A}^{s'}_\Omega)_{>h} \end{aligned}$$

with the third line following from Proposition 2.4.2. The two summands in the third line naturally form the components of a slope-$\leq h$ decomposition, so passing to homology yields the first sentence of the proposition, and the second sentence then follows immediately from Proposition 2.4.3. □

We're now in a position to prove the subtler cohomology analogue of Proposition 2.4.4.

Proposition 2.4.5. *Given a slope datum (U_t, Ω, h) and a rigid Zariski-closed subset $\Sigma \subset \Omega$, the complex $C^\bullet(K^p, \mathcal{D}_\Sigma)$ and the cohomology module $H^*(K^p, \mathcal{D}_\Sigma)$ admit slope-$\leq h$ decompositions, and there is an isomorphism*

$$H^*(K^p, \mathcal{D}_\Sigma)_{\leq h} \simeq H^*(K^p, \mathbf{D}^s_\Sigma)_{\leq h}$$

for any $s \geq s[\Omega]$. Furthermore, given an affinoid subdomain $\Omega' \subset \Omega$, there are canonical isomorphisms

$$C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega') \simeq C^\bullet(K^p, \mathcal{D}_{\Omega'})_{\leq h}$$

and

$$H^*(K^p, \mathcal{D}_\Omega)_{\le h} \otimes_{A(\Omega)} A(\Omega') \simeq H^*(K^p, \mathcal{D}_{\Omega'})_{\le h}.$$

Proof. By a topological version of the duality stated in the final paragraph of §2.2, we have a natural isomorphism

$$\begin{array}{rcl} C^\bullet(K^p, \mathbf{D}^s_\Sigma) & = & C^\bullet(K^p, \mathcal{L}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Sigma))) \\ & \simeq & \mathcal{L}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega), A(\Sigma)) \end{array}$$

for any $s \ge s[\Omega]$. By assumption, $C_\bullet(K^p, \mathbf{A}^s_\Omega)$ admits a slope-$\le h$ decomposition, so we calculate

$$\begin{array}{rcl} C^\bullet(K^p, \mathbf{D}^s_\Sigma) & \simeq & \mathcal{L}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega), A(\Sigma)) \\ & \simeq & \mathcal{L}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \\ & & \oplus \mathcal{L}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{>h}, A(\Sigma)). \end{array}$$

By Proposition 2.4.2, passing to the inverse limit over s in this isomorphism yields a slope-$\le h$ decomposition of $C^\bullet(K^p, \mathcal{D}_\Sigma)$ together with a natural isomorphism

$$C^\bullet(K^p, \mathcal{D}_\Sigma)_{\le h} \simeq C^\bullet(K^p, \mathbf{D}^s_\Sigma)_{\le h} \simeq \mathcal{L}_{A(\Omega)}(C^\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Sigma))$$

for any $s \ge s[\Omega]$. This proves the first sentence of the proposition.

For the second sentence, we first note that since $C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}$ is a complex of finite $A(\Omega)$-modules, the natural map

$$\mathcal{L}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Omega)) \to \mathrm{Hom}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Omega))$$

is an isomorphism by Lemma 2.2 of [Buz07]. Next, note that if R is a commutative ring, S is a flat R-algebra, and M, N are R-modules with M finitely presented, the natural map $\mathrm{Hom}_R(M, N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$ is an isomorphism. With these two facts in hand, we calculate as follows:

$$\begin{array}{rcl} C^\bullet(K^p, \mathcal{D}_\Omega)_{\le h} \otimes_{A(\Omega)} A(\Omega') & \simeq & \mathrm{Hom}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Omega)) \otimes_{A(\Omega)} A(\Omega') \\ & \simeq & \mathrm{Hom}_{A(\Omega')}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h} \otimes_{A(\Omega)} A(\Omega'), A(\Omega')) \\ & \simeq & \mathrm{Hom}_{A(\Omega')}(C_\bullet(K^p, \mathbf{A}^s_{\Omega'})_{\le h}, A(\Omega')) \\ & \simeq & C^\bullet(K^p, \mathcal{D}_{\Omega'})_{\le h}, \end{array}$$

where the third line follows from Proposition 2.3.3. Passing to cohomology, the result follows as in the proof of Proposition 2.4.3.

Recall from the end of §2.1 the alternate module of distributions $\tilde{\mathcal{D}}_\Omega$. For completeness's sake, we sketch the following result.

Proposition 2.4.6. *If the complexes $C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)$ and $C^\bullet(K^p, \mathcal{D}_\Omega)$ both admit slope-$\le h$ decompositions for the $\tilde{U}_t$-action, the natural map $\tilde{\mathcal{D}}_\Omega \to \mathcal{D}_\Omega$ induces an isomorphism*

$$H^*(K^p, \tilde{\mathcal{D}}_\Omega)_{\le h} \simeq H^*(K^p, \mathcal{D}_\Omega)_{\le h}.$$

Proof. The morphism $f : \tilde{\mathcal{D}}_\Omega \to \mathcal{D}_\Omega$ induces a morphism $\phi : C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\le h} \to C^\bullet(K^p, \mathcal{D}_\Omega)_{\le h}$. Choose an arbitrary weight $\lambda \in \Omega(\overline{\mathbf{Q}_p})$; since

$$\mathcal{D}_\Omega \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda \simeq \tilde{\mathcal{D}}_\Omega \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda \simeq \mathcal{D}_\lambda,$$

we calculate

$$\begin{aligned} C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\le h} \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda &\simeq C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda)_{\le h} \\ &\simeq C^\bullet(K^p, \mathcal{D}_\lambda)_{\le h} \\ &\simeq C^\bullet(K^p, \mathcal{D}_\Omega \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda)_{\le h} \\ &\simeq C^\bullet(K^p, \mathcal{D}_\Omega)_{\le h} \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda, \end{aligned}$$

where the first and fourth lines follow by Proposition 2.3.3. Thus $\phi \bmod \mathfrak{m}_\lambda$ is an isomorphism in the category of $A(\Omega)/\mathfrak{m}_\lambda$-module complexes. Writing $C^\bullet(\phi)$ for the cone of ϕ, we have an exact triangle

$$C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\le h} \xrightarrow{\phi} C^\bullet(K^p, \mathcal{D}_\Omega)_{\le h} \to C^\bullet(\phi) \to C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\le h}[1]$$

in $\mathbf{D}^b(A(\Omega))$. Since $C^\bullet(K^p, \tilde{\mathcal{D}}_\Omega)_{\le h}$ and $C^\bullet(K^p, \mathcal{D}_\Omega)_{\le h}$ are complexes of projective $A(\Omega)$-modules, the cone $C^\bullet(\phi)$ is a complex of projective modules as well; therefore, the functors $- \otimes^{\mathbf{L}}_{A(\Omega)} A(\Omega)/\mathfrak{m}$ and $- \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}$ agree on these three complexes (cf. [Wei94], 10.6). In particular, applying $- \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda$ yields an exact triangle

$$C^\bullet(K^p, \mathcal{D}_\lambda)_{\le h} \xrightarrow{\phi \bmod \mathfrak{m}_\lambda} C^\bullet(K^p, \mathcal{D}_\lambda)_{\le h} \to C^\bullet(\phi) \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda \to C^\bullet(K^p, \mathcal{D}_\lambda)_{\le h}[1]$$

in $\mathbf{D}^b(A(\Omega)/\mathfrak{m}_\lambda)$. Since $\phi \bmod \mathfrak{m}_\lambda$ is an isomorphism, $C^\bullet(\phi) \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda$ is acyclic, which in turn implies that $\mathfrak{m}_\lambda$ is not contained in the support of $H^*(C^\bullet(\phi))$. But $\mathfrak{m}_\lambda$ is an arbitrary maximal ideal in $A(\Omega)$, so Nakayama's lemma now shows that $H^*(C^\bullet(\phi))$ vanishes identically. Thus $C^\bullet(\phi)$ is acyclic and ϕ is a quasi-isomorphism as desired. □

We now recall a fundamental theorem of Ash-Stevens and Urban (Theorem 6.4.1 of [AS08], Proposition 4.3.10 of [Urb11]) relating overconvergent cohomology classes of small slope to classical automorphic forms. The possibility of such a result was largely the original *raison d'etre* of overconvergent cohomology. For any weight $\lambda \in \mathcal{W}(\overline{\mathbf{Q}_p})$ we define the *finite slope subspace* $H^*(K^p, \mathcal{D}_\lambda)_{\mathrm{fs}}$ as the intersection $\cap_{n \ge 1} \mathrm{im} U_t^n | H^*(K^p, \mathcal{D}_\lambda)$, where U_t is any controlling operator. This definition is independent of the choice of t. Now suppose λ is an arithmetic weight, and let k be the finite Galois extension of $\mathbf{Q}_p$ generated by the values of λ. We write $\lambda = \lambda^{\mathrm{alg}}\varepsilon$ where $\lambda^{\mathrm{alg}} \in X^*_+$ and $\varepsilon : T(\mathbf{Z}_p) \to k^\times$ is a finite order character. Given $w \in W$, write ${}^w\lambda^{\mathrm{alg}}$ for the character $w \cdot (\lambda^{\mathrm{alg}} + \rho) - \rho$. Let $\mathbf{T}_{\lambda,h}$ be the subalgebra of $\mathrm{End}_k(H^*(K^p, \mathcal{D}_\lambda)_{\le h})$ generated by $\mathbf{T}(K^p)$, and let $\mathbf{T}_\lambda$ be the projective limit $\lim_{\infty \leftarrow h} \mathbf{T}_{\lambda,h}$. If $\phi : \mathbf{T}_\lambda \to \overline{\mathbf{Q}_p}$ is an eigenpacket, the semigroup character

$$\begin{aligned} \mu_\phi : \Lambda^+ &\to \mathbf{Q}_{>0} \\ t &\mapsto v_p(\phi(U_t)) \end{aligned}$$

extends uniquely to a character $\mu_\phi : T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \to \mathbf{Q}$.

Definition 2.4.7. *Fix a controlling operator U_t, $t \in \Lambda$. Given an arithmetic weight $\lambda = \lambda^{\mathrm{alg}}\varepsilon$, a rational number h is a* small slope *for λ if*

$$h < \inf_{w \in W \setminus \{\mathrm{id}\}} v_p({}^w\lambda^{\mathrm{alg}}(t)) - v_p(\lambda^{\mathrm{alg}}(t)).$$

If $\phi : \mathbf{T}_\lambda \to \overline{\mathbf{Q}_p}$ is an eigenpacket, ϕ is numerically non-critical *if*

$$v_p(\mu_\phi(t)) < \inf_{w \in W \smallsetminus \{id\}} v_p({}^w\lambda^{\mathrm{alg}}(t)) - v_p(\lambda^{\mathrm{alg}}(t))$$

for all $t = \mu^\vee(p^{-1})$ with $\mu^\vee \in X_*(T)$ a positive coroot.

Theorem 2.4.8 (Ash-Stevens, Urban). *Fix an arithmetic weight $\lambda = \lambda^{\mathrm{alg}}\varepsilon$.*

a) Fix a controlling operator U_t. If h is a small slope for λ, there is a natural isomorphism of Hecke modules

$$H^*(K^p, \mathcal{D}_\lambda)_{\le h} \simeq H^*(Y(K^p I_1^c), V_{\lambda^{\mathrm{alg}}}(k))_{\le h}^{T(\mathbf{Z}/p^c\mathbf{Z})=\epsilon}$$

for any $c \ge c(\varepsilon)$.

b) If $\phi \in H^*(K^p, \mathcal{D}_\lambda)_{\mathrm{fs}}$ is a numerically non-critical eigenclass, the morphism

$$H^*(K^p, \mathcal{D}_\lambda) \to H^*(Y(K^p I_1^c), V_{\lambda^{\mathrm{alg}}}(k))$$

is a Hecke-equivariant isomorphism on the ϕ-generalized eigenspace for any $c \ge c(\varepsilon)$.

More generally, suppose $\phi : \mathbf{T}(K^p) \to \overline{\mathbf{Q}_p}$ is an eigenpacket associated with a classical algebraic automorphic representation of weight λ^{alg}. We say ϕ has *finite slope* if $\phi(U_t) \neq 0$ for some controlling operator. Choosing $h \ge v_p(\phi(U_t))$, we define ϕ to be *non-critical* if the map

$$H^*(K^p, \mathcal{D}_\lambda)_{\le h} \to H^*(Y(K^p I_1^c), V_{\lambda^{\mathrm{alg}}}(k))_{\le h}$$

induces a Hecke-equivariant isomorphism on the ϕ-generalized eigenspaces. According to Theorem 2.4.8, every numerically non-critical eigenpacket is non-critical. However, all known examples point to the suspicion that numerically critical cuspidal eigenpackets are typically non-critical, with the critical cuspidal eigenpackets rising via functorial maps from smaller groups.

3 Proofs of the main results

Fix a choice of a tame level K^p and an augmented Borel-Serre complex $C_\bullet(K^p, -)$.

3.1 Proof of Theorem 1.1

The spectral sequences Fix a rigid Zariski closed subset $\Sigma \subset \Omega$. By the isomorphisms proved in §2.4, it suffices to construct a spectral sequence $\mathrm{Ext}^i_{A(\Omega)}(H_j(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathbf{D}^s_\Sigma)_{\le h}$ for some $s \ge s[\Omega]$. From Proposition 2.4.5, we have a natural isomorphism

$$\mathrm{Hom}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \simeq C^\bullet(K^p, \mathbf{D}^s_\Sigma)_{\le h}.$$

Let $\iota : C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h} \hookrightarrow C_\bullet(K^p, \mathbf{A}^s_\Omega)$ denote the canonical inclusion, and $\pi : C_\bullet(K^p, \mathbf{A}^s_\Omega) \twoheadrightarrow C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}$ the canonical projection. The formula $T \to \pi \circ \tilde{\xi}(T) \circ \iota$ defines an algebra homomorphism $\mathbf{T}(K^p) \to \mathrm{End}_{\mathbf{K}(A(\Omega))}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h})$ which induces the usual Hecke action on $H_*(K^p, \mathbf{A}^s_\Omega)_{\le h}$; making the analogous definition on $C^\bullet(K^p, \mathbf{D}^s_\Sigma)_{\le h}$, the above isomorphism of complexes upgrades to an isomorphism of $\mathbf{T}(K^p)$-module complexes in $\mathbf{K}^b(A(\Omega))$. Since $C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}$ is a complex of projective $A(\Omega)$-modules, this isomorphism in turn induces an isomorphism

$$\mathbf{R}\mathrm{Hom}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \simeq C^\bullet(K^p, \mathbf{D}^s_\Sigma)_{\le h}$$

of $\mathbf{T}(K^p)$-module complexes in $\mathbf{D}^b(A(\Omega))$ (cf. Theorem 10.7.4 of [Wei94]). Passing to cohomology yields

$$\mathbf{E}\mathrm{xt}^i_{A(\Omega)}(C_\bullet(K^p,\mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \simeq H^i(K^p,\mathbf{D}^s_\Sigma)_{\le h}.$$

Quite generally, if $C_\bullet$ is a chain complex of R-modules equipped with an algebra homomorphism $\varphi : S \to \mathrm{End}_{\mathbf{K}(R)}(C_\bullet)$ and N is any R-module, φ induces a natural S-module structure on the homology groups $H_*(C_\bullet)$ and the hyperext groups $\mathbf{E}\mathrm{xt}^*_R(C_\bullet, N)$, and the hyperext spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R(H_j(A_\bullet), N) \Rightarrow \mathbf{E}\mathrm{xt}^{i+j}_R(A_\bullet, N)$$

is a spectral sequence of S-modules. The result follows.

For the Tor spectral sequence, the isomorphism

$$C^\bullet(K^p,\mathbf{D}^s_\Omega)_{\le h} \otimes_{A(\Omega)} A(\Sigma) \simeq C^\bullet(K^p,\mathbf{D}^s_\Sigma)_{\le h}$$

yields an isomorphism

$$C^\bullet(K^p,\mathbf{D}^s_\Omega)_{\le h} \otimes^{\mathbf{L}}_{A(\Omega)} A(\Sigma) \simeq C^\bullet(K^p,\mathbf{D}^s_\Sigma)_{\le h}$$

of $\mathbf{T}(K^p)$-module complexes in $\mathbf{D}^b(A(\Omega))$, and the result follows analogously from the hypertor spectral sequence

$$\mathrm{Tor}^R_{-i}(H^j(C^\bullet), N) \Rightarrow \mathbf{T}\mathrm{or}^R_{-i-j}(C^\bullet, N).$$

Remark 3.1.1. If (Ω, h) is a slope datum, Σ_1 is Zariski-closed in Ω, and Σ_2 is Zariski-closed in Σ_1, the transitivity of the derived tensor product yields an isomorphism

$$\begin{aligned} C^\bullet(K^p,\mathcal{D}_{\Sigma_2})_{\le h} &\simeq C^\bullet(K^p,\mathcal{D}_\Omega)_{\le h} \otimes^{\mathbf{L}}_{A(\Omega)} A(\Sigma_2) \\ &\simeq C^\bullet(K^p,\mathcal{D}_\Omega)_{\le h} \otimes^{\mathbf{L}}_{A(\Omega)} A(\Sigma_1) \otimes^{\mathbf{L}}_{A(\Sigma_1)} A(\Sigma_2) \\ &\simeq C^\bullet(K^p,\mathcal{D}_{\Sigma_1})_{\le h} \otimes^{\mathbf{L}}_{A(\Sigma_1)} A(\Sigma_2) \end{aligned}$$

which induces a relative version of the Tor spectral sequence, namely

$$E_2^{i,j} = \mathrm{Tor}^i_{A(\Sigma_1)}(H^j(K^p,\mathcal{D}_{\Sigma_1})_{\le h}, A(\Sigma_2)) \Rightarrow H^{i+j}(K^p,\mathcal{D}_{\Sigma_2})_{\le h}.$$

This spectral sequence plays an important role in Newton's Appendix.

The boundary and Borel-Moore/compactly supported spectral sequences Recall the complex $C_\bullet(K^p,-)$ was defined by choosing a triangulation of the Borel-Serre compactification $\overline{Y(K^pI)}$ of $Y(K^pI)$. The induced triangulation of the boundary yields a complex $C^\partial_\bullet(K^p,-)$ which calculates $H_*(\partial\overline{Y(K^pI)}, M)$, together with a morphism $\phi : C^\partial_\bullet(K^p,-) \to C_\bullet(K^p,-)$ inducing the usual morphism on homology. The boundary and Borel-Moore/compactly supported sequences, and the morphisms between them, follow from beholding the diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{A(\Omega)}(C^{\mathrm{BM}}_\bullet(K^p,\mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \arrow[r] \arrow[d] & C^\bullet_c(K^p,\mathbf{D}^s_\Sigma)_{\le h} \arrow[d] \\
\mathbf{R}\mathrm{Hom}_{A(\Omega)}(C_\bullet(K^p,\mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \arrow[r] \arrow[d] & C^\bullet(K^p,\mathbf{D}^s_\Sigma)_{\le h} \arrow[d] \\
\mathbf{R}\mathrm{Hom}_{A(\Omega)}(C^\partial_\bullet(K^p,\mathbf{A}^s_\Omega)_{\le h}, A(\Sigma)) \arrow[r] & C^\bullet_\partial(K^p,\mathbf{D}^s_\Sigma)_{\le h}
\end{tikzcd}$$

in which the horizontal arrows are quasi-isomorphisms, the columns are exact triangles in $\mathbf{D}^b(A(\Omega))$, and the diagram commutes up to homotopy for the natural action of $\mathbf{T}(K^p)$.

3.2 A construction of eigenvarieties

In this section we briefly sketch the construction of the rigid spaces $\mathscr{X}(K^p)$ described in Theorem 1.3.

Step One: the spectral variety and its covering. Fix a controlling operator $U = U_t$. Given $\Omega \subset \mathcal{W}$ an admissible open affinoid and $s \geq s[\Omega]$, let $F_\Omega(T) = \sum_{n=0}^\infty a_{n,\Omega} T^n \in A(\Omega)\{\{T\}\}$ be the characteristic power series of $\tilde{U}$ acting on the complex $C_\bullet(K^p, \mathbf{A}_\Omega^s)$. Suppose $\Omega' \subset \Omega$; since $F_\Omega(T)|_{\Omega'} = F_{\Omega'}(T)$, a simple calculation gives $a_{n,\Omega}|_{\Omega'} = a_{n,\Omega'}$. By Tate's acyclicity theorem there exist unique elements $a_n \in \mathcal{O}(\mathcal{W})$ such that $F(T) = \sum_{n=0}^\infty a_n T^n \in \mathcal{O}(\mathcal{W})\{\{T\}\}$ restricts to $F_\Omega(T)$ on any admissible affinoid open subset $\Omega \subset \mathcal{W}$. The zero locus of $F(T)$ cuts out a Fredholm hypersurface $\mathcal{Z} \subset \mathcal{W} \times \mathbf{A}^1$, and we regard $\mathcal{Z}$ as a locally G-ringed space in the usual way (cf. [Con99] for a thorough treatment of Fredholm hypersurfaces). Given a connected affinoid $\Omega \subset \mathcal{W}$ and a rational $h \in \mathbf{Q}_{\geq 0}$, define the admissible open

$$\mathcal{Z}_{\Omega,h} = \mathcal{Z} \cap \left\{(\omega, z), \omega \in \Omega, z \in \mathbf{A}^1 \text{ with } |z| \leq p^h\right\};$$

the coordinate ring is naturally $A(\mathcal{Z}_{\Omega,h}) = A(\Omega)\left\langle p^h T\right\rangle / (F_\Omega(T))$. The morphism $\mathcal{Z}_{\Omega,h} \to \Omega$ is flat but not necessarily finite. Let $\mathscr{C}$ov be the set of $\mathcal{Z}_{\Omega,h}$'s such that $\mathcal{Z}_{\Omega,h} \to \Omega$ is *finite flat*. The significance of the finiteness condition is that $\mathcal{Z}_{\Omega,h}$ is finite if and only if $C_\bullet(K^p, \mathbf{A}_\Omega^s)$ admits a slope-$\leq h$ decomposition: $\mathcal{Z}_{\Omega,h}$ is finite flat if and only if it is disconnected from its complement in $\mathcal{Z}_{\Omega,\infty}$, if and only if F_Ω admits a slope-$\leq h$ factorization $F_\Omega = Q_{\Omega,h} \cdot R_{\Omega,h}$, in which case $A(\mathcal{Z}_{\Omega,h}) \simeq A(\Omega)[T]/Q_{\Omega,h}(T)$. Since $C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h}$ is the kernel of $Q^*(\tilde{U})$, and $\tilde{U}$ acts invertibly on $C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h}$, the map

$$\begin{array}{rcl} A(\Omega)[T] & \to & \mathrm{End}_{A(\Omega)}(C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h}) \\ T & \mapsto & \tilde{U}^{-1} \end{array}$$

puts a canonical $A(\mathcal{Z}_{\Omega,h})$-module structure on $C_\bullet(K^p, \mathbf{A}_\Omega^s)_{\leq h}$; by the isomorphisms proven in §2.4, this map induces a canonical $A(\mathcal{Z}_{\Omega,h})$-module structure on $C_\bullet(K^p, \mathcal{A}_\Omega)_{\leq h}$ and on $C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$. Note that by Proposition 2.3.2, *every* point $z \in \mathcal{Z}$ is contained in some $\mathcal{Z}_{\Omega,h}$ with $\mathcal{Z}_{\Omega,h} \in \mathscr{C}$ov. By a fundamental theorem of Buzzard (Lemma 4.5 and Theorem 4.6 of [Buz07]), the elements of $\mathscr{C}$ov form an admissible covering of $\mathcal{Z}$.

Step Two: the cocycle condition. If $\mathcal{Z}_{\Omega,h} \in \mathscr{C}$ov and $\Omega' \subset \Omega$ with Ω' connected, a moment's thought yields $A(\mathcal{Z}_{\Omega',h}) \simeq A(\mathcal{Z}_{\Omega,h}) \otimes_{A(\Omega)} A(\Omega')$, so then $\mathcal{Z}_{\Omega',h} \in \mathscr{C}$ov. Fix $\mathcal{Z}_{\Omega',h'} \subseteq \mathcal{Z}_{\Omega,h} \in \mathscr{C}$ov with $\mathcal{Z}_{\Omega',h'} \in \mathscr{C}$ov; we necessarily have $\Omega' \subseteq \Omega$, and we may assume $h' \leq h$. Set $C_{\Omega,h} = C^\bullet(K^p, \mathcal{D}_\Omega)_{\leq h}$. We now trace through the following sequence of canonical isomorphisms:

$$\begin{array}{rcl} C_{\Omega,h} \otimes_{A(\mathcal{Z}_{\Omega,h})} A(\mathcal{Z}_{\Omega',h'}) & \simeq & C_{\Omega,h} \otimes_{A(\mathcal{Z}_{\Omega,h})} A(\mathcal{Z}_{\Omega',h}) \otimes_{A(\mathcal{Z}_{\Omega',h})} A(\mathcal{Z}_{\Omega',h'}) \\ & \simeq & \left(C_{\Omega,h} \otimes_{A(\mathcal{Z}_{\Omega,h})} A(\mathcal{Z}_{\Omega,h}) \otimes_{A(\Omega)} A(\Omega')\right) \otimes_{A(\mathcal{Z}_{\Omega',h})} A(\mathcal{Z}_{\Omega',h'}) \\ & \simeq & \left(C_{\Omega,h} \otimes_{A(\Omega)} A(\Omega')\right) \otimes_{A(\mathcal{Z}_{\Omega',h})} A(\mathcal{Z}_{\Omega',h'}) \\ & \simeq & C_{\Omega',h} \otimes_{A(\mathcal{Z}_{\Omega',h})} A(\mathcal{Z}_{\Omega',h'}) \\ & \simeq & C_{\Omega',h'}. \end{array}$$

The fourth line here follows from Proposition 2.4.5.

Step Three: coherent $\mathcal{O}$-modules. Given $\mathcal{Z}_{\Omega',h'} \subseteq \mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$ with $\mathcal{Z}_{\Omega',h'} \in \mathscr{C}\mathrm{ov}$, $\mathcal{Z}_{\Omega',h'}$ is an affinoid subdomain of $\mathcal{Z}_{\Omega,h}$, so $A(\mathcal{Z}_{\Omega',h'})$ is $A(\mathcal{Z}_{\Omega,h})$-flat. In particular, the functor $-\otimes_{A(\mathcal{Z}_{\Omega,h})} A(\mathcal{Z}_{\Omega',h'})$ commutes with taking cohomology of any complex of $A(\mathcal{Z}_{\Omega,h})$-modules. Hence, taking cohomology in the isomorphism of step three yields a canonical isomorphism

$$H^*(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{A(\mathcal{Z}_{\Omega,h})} A(\mathcal{Z}_{\Omega',h'}) \simeq H^*(K^p, \mathcal{D}_{\Omega'})_{\leq h'}.$$

This is exactly what we need in order to verify that the assignments

$$\mathcal{Z}_{\Omega,h} \mapsto H^*(K^p, \mathcal{D}_\Omega)_{\leq h},\ \mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$$

glue together into a coherent $\mathcal{O}_{\mathcal{Z}}$-module sheaf $\mathcal{M}^* = \oplus \mathcal{M}^n$ such that $\mathcal{M}^n(\mathcal{Z}_{\Omega,h}) \simeq H^n(K^p, \mathcal{D}_\Omega)_{\leq h}$ for $\mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$. Now, for $\mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$ let $\mathbf{T}_{\Omega,h}$ be the commutative subalgebra of $\mathrm{End}_{A(\mathcal{Z}_{\Omega,h})}(H^*(K^p, \mathcal{D}_\Omega)_{\leq h})$ generated by $\mathbf{T}(K^p) \otimes_{\mathbf{Q}_p} A(\Omega)$, with

$$\psi_{\Omega,h} : \mathbf{T}(K^p) \otimes_{\mathbf{Q}_p} A(\Omega) \to \mathbf{T}_{\Omega,h}$$

the structure map. For any Noetherian ring A, finite A-module M, and flat A-algebra B, there is a canonical isomorphism $\mathrm{End}_A(M) \otimes_A B \simeq \mathrm{End}_B(M \otimes_A B)$. In particular, we immediately obtain canonical isomorphisms

$$\mathbf{T}_{\Omega,h} \otimes_{A(\mathcal{Z}_{\Omega,h})} A(\mathcal{Z}_{\Omega',h'}) \simeq \mathbf{T}_{\Omega',h'},$$

whereby the $A(\mathcal{Z}_{\Omega,h})$-algebras $\mathbf{T}_{\Omega,h}$ glue together into a coherent sheaf $\mathcal{T}$ of $\mathcal{O}_{\mathcal{Z}}$-algebras. By an obvious rigid version of the "relative Spec" construction (see e.g. §2.2 of [Con06]) the affinoid rigid spaces $\mathscr{X}_{\Omega,h} = \mathrm{Sp}\mathbf{T}_{\Omega,h}$ glue together into a rigid space $\mathscr{X}$, and the natural morphisms

$$w : \mathscr{X}_{\Omega,h} \to \mathcal{Z}_{\Omega,h} \to \Omega$$

glue into a morphism $w : \mathscr{X} \to \mathcal{W}$. Since $\mathcal{Z}$ is separated and $\mathscr{X}$ is finite over $\mathcal{Z}$, $\mathscr{X}$ is separated. Each $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ is naturally a finite $\mathbf{T}_{\Omega,h}$-module; since

$$\begin{aligned} H^*(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega',h'} &= H^*(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,h} \otimes_{A(\mathcal{Z}_{\Omega,h})} A(\mathcal{Z}_{\Omega',h'}) \\ &\simeq H^*(K^p, \mathcal{D}_{\Omega'})_{\leq h'}, \end{aligned}$$

the cohomology groups $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ glue together into a sheaf of coherent $\mathcal{O}_{\mathscr{X}}$-modules. For any fixed element $T \in \mathbf{T}(K^p)$, the sections $\psi_{\Omega,h}(T \otimes 1) \in \mathcal{O}(\mathscr{X}_{\Omega,h})$ glue to a unique global section $\psi(T) \in \mathcal{O}(\mathscr{X})$, and ψ is easily seen to be an algebra homomorphism. Power-boundedness of $\psi(T)$ follows immediately from the fact that any element $\gamma \in \Delta$ acts on $\mathcal{D}_\Omega$ with operator norm at most one for the family of norms defining the Fréchet topology on $\mathcal{D}_\Omega$. At this point, claims i., iii., and iv. of Theorem 1.5 are immediate consequences of our construction. If $x \in \mathscr{X}(\overline{\mathbf{Q}_p})$, choose some $\mathscr{X}_{\Omega,h}$ with $w(x) \in \Omega$ and $h > v_p(\psi(x)(U_t))$; writing $\mathfrak{M}_x$ for the maximal ideal of $\mathbf{T}_{\Omega,h}$ determined by x, claim ii. follows immediately from setting $\Sigma = w(x)$ in the Tor spectral sequence and then localizing the spectral sequence at $\mathfrak{M}_x$. Claim v. is straightforward, though slightly tedious, and we point the reader to [Han12b] for details.

3.3 The support of overconvergent cohomology

Let R be a Noetherian ring, and let M be a finite R-module. We say M has *full support* if $\mathrm{Supp}(M) = \mathrm{Spec}(R)$, and that M is *torsion* if $\mathrm{ann}(M) \neq 0$. We shall repeatedly use the following basic result.

Proposition 3.3.1. *If* $\mathrm{Spec}(R)$ *is reduced and irreducible, the following are equivalent:*
i) M *is faithful (i.e.* $\mathrm{ann}(M) = 0$*),*
ii) M *has full support,*
iii) M *has nonempty open support,*
iv) $\mathrm{Hom}_R(M, R) \neq 0$,
v) $M \otimes_R K \neq 0$, $K = \mathrm{Frac}(R)$.

Proof. Since M is finite, $\mathrm{Supp}(M)$ is the underlying topological space of $\mathrm{Spec}(R/\mathrm{ann}(M))$, so i) obviously implies ii). If $\mathrm{Spec}(R/\mathrm{ann}(M)) = \mathrm{Spec}(R)$ as topological spaces, then $\mathrm{ann}(M) \subset \sqrt{(0)} = (0)$ since R is reduced, so ii) implies i). The set $\mathrm{Supp}(M) = \mathrm{Spec}(R/\mathrm{ann}(M))$ is *a priori* closed; since $\mathrm{Spec}(R)$ is irreducible by assumption, the only nonempty simultaneously open and closed subset of $\mathrm{Spec}(R)$ is all of $\mathrm{Spec}(R)$, so ii) and iii) are equivalent. By finiteness, M has full support if and only if (0) is an associated prime of M, if and only if there is an injection $R \hookrightarrow M$; tensoring with K implies the equivalence of ii) and v). Finally, $\mathrm{Hom}_R(M, R) \otimes_R K \simeq \mathrm{Hom}_K(M \otimes_R K, K)$, so $M \otimes_R K \neq 0$ if and only if $\mathrm{Hom}_R(M, R) \neq 0$, whence iv) and v) are equivalent. □

Proof of Theorem 1.2.i. (I'm very grateful to Jack Thorne for suggesting this proof.) Tensoring the Ext spectral sequence with $K(\Omega) = \mathrm{Frac}(A(\Omega))$, it degenerates to isomorphisms

$$\mathrm{Hom}_{K(\Omega)}(H_i(K^p, \mathcal{A}_\Omega)_{\leq h} \otimes_{A(\Omega)} K(\Omega), K(\Omega)) \simeq H^i(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{A(\Omega)} K(\Omega),$$

so the claim is immediate from the preceding proposition. □

Proof of Theorem 1.2.ii. We give the proof in two steps, with the first step naturally breaking into two cases. In the first step, we prove the result assuming Ω contains an arithmetic weight. In the second step, we eliminate this assumption via analytic continuation.

Step One, Case One: G doesn't have a discrete series. Let $\mathcal{W}^{\mathrm{sd}}$ be the rigid Zariski closure in $\mathcal{W}$ of the arithmetic weights whose algebraic parts are the highest weights of irreducible G-representations with nonvanishing $(\mathfrak{g}, K_\infty)$-cohomology. A simple calculation using §II.6 of [BW00] shows that $\mathcal{W}^{\mathrm{sd}}$ is the union of its countable set of irreducible components, each of dimension $r(G)$. An arithmetic weight is *non-self-dual* if $\lambda \notin \mathcal{W}^{\mathrm{sd}}$.

Now, by assumption Ω contains an arithmetic weight, so Ω automatically contains a Zariski dense set $\mathcal{N}_h \subset \Omega \smallsetminus \Omega \cap \mathcal{W}^{\mathrm{sd}}$ of non-self-dual arithmetic weights for which h is a small slope. By Theorem 2.4.8 together with Matsushima's formula, $H^*(K^p, \mathcal{D}_\lambda)_{\leq h}$ vanishes identically for any $\lambda \in \mathcal{N}_h$. For any fixed $\lambda \in \mathcal{N}_h$, suppose $\mathfrak{m}_\lambda \in \mathrm{Supp}_\Omega H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$; let d be the largest integer with $\mathfrak{m}_\lambda \in \mathrm{Supp}_\Omega H^d(K^p, \mathcal{D}_\Omega)_{\leq h}$. Taking $\Sigma = \lambda$ in the Tor spectral sequence gives

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{A(\Omega)}(H^j(K^p, \mathcal{D}_\Omega)_{\leq h}, A(\Omega)/\mathfrak{m}_\lambda) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\lambda)_{\leq h}.$$

The entry $E_2^{0,d} = H^d(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda$ is nonzero by Nakayama's lemma, and is stable since every row of the E_2-page above the dth row vanishes by assumption. In particular, $E_2^{0,d}$ contributes a nonzero summand to the grading on $H^d(K^p, \mathcal{D}_\lambda)_{\leq h}$ - but this module is zero, contradicting our assumption that $\mathfrak{m}_\lambda \in \mathrm{Supp}_\Omega H^*(K^p, \mathcal{D}_\Omega)$. Therefore, $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ does *not* have full support, so is not a faithful $A(\Omega)$-module.

Step One, Case Two: G has a discrete series. The idea is the same as Case One, but with $\mathcal{N}_h$ replaced by $\mathcal{R}_h$, the set of arithmetic weights with *regular* algebraic part for which h is a small slope. For these weights, Proposition 2.4.8 and Matsushima's formula together with known results on $(\mathfrak{g}, K_\infty)$-cohomology (see e.g. Sections 4-5 of [LS04]) implies that $H^i(K^p, \mathcal{D}_\lambda)_{\leq h}$, $\lambda \in \mathcal{R}_h$ vanishes for $i \neq d_G = \frac{1}{2}\mathrm{dim} G(\mathbf{R})/Z_\infty K_\infty$. The Tor spectral sequence with $\Sigma = \lambda \in \mathcal{R}_h$ then shows that $\mathcal{R}_h$ doesn't meet $\mathrm{Supp}_\Omega H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ for any $i > d_G$. The Ext spectral sequence with $\Sigma = \lambda \in \mathcal{R}_h$ then shows that $\mathcal{R}_h$ doesn't meet $\mathrm{Supp}_\Omega H_i(K^p, \mathcal{A}_\Omega)_{\leq h}$ for any $i < d_G$, whence the Ext spectral sequence with $\Sigma = \Omega$ shows that $\mathcal{R}_h$ doesn't meet $\mathrm{Supp}_\Omega H^i(K^p, \mathcal{D}_\Omega)_{\leq h}$ for any $i < d_G$. The result follows.

Step Two. We maintain the notation of §3.2. As in that subsection, $H^n(K^p, \mathcal{D}_\Omega)_{\leq h}$ glues together over the affinoids $\mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$ into a coherent $\mathcal{O}_\mathcal{Z}$-module sheaf $\mathcal{M}^n$, and in particular, the support of $\mathcal{M}^n$ is a closed analytic subset of $\mathcal{Z}$. Let $\pi : \mathcal{Z} \to \mathcal{W}$ denote the natural projection. For any $\mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$, we have

$$\pi_* \mathrm{Supp}_{\mathcal{Z}_{\Omega,h}} \mathcal{M}^n(\mathcal{Z}_{\Omega,h}) = \mathrm{Supp}_\Omega H^n(K^p, \mathcal{D}_\Omega)_{\leq h}.$$

Suppose $\mathrm{Supp}_\Omega H^n(K^p, \mathcal{D}_\Omega)_{\leq h} = \Omega$ for some $\mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$. This implies that $\mathrm{Supp}_{\mathcal{Z}_{\Omega,h}} \mathcal{M}^n(\mathcal{Z}_{\Omega,h})$ contains a closed subset of dimension equal to $\mathrm{dim}\mathcal{Z}$, so contains an irreducible component of $\mathcal{Z}_{\Omega,h}$. Any irreducible component of $\mathcal{Z}_{\Omega,h}$ is an admissible open affinoid in $\mathcal{Z}$, so this in turn implies that $\mathrm{Supp}_\mathcal{Z} \mathcal{M}^n$ contains an affinoid open. Since $\mathrm{Supp}_\mathcal{Z} \mathcal{M}^n$ is a priori closed, Corollary 2.2.6 of [Con99] implies that $\mathrm{Supp}_\mathcal{Z} \mathcal{M}^n$ contains an entire irreducible component of $\mathcal{Z}$, say $\mathcal{Z}_0$. The irreducible component $\mathcal{Z}_0$ corresponds, by Theorem 4.3.2 of [Con99], to a nonconstant irreducible Fredholm series

$$F_0(T) = 1 + \sum_{j=1}^{\infty} \omega_j T^j,\ \omega_j \in \mathcal{O}(\mathcal{W})$$

dividing $F(T)$. I claim the image of $\mathcal{Z}_0$ under π is Zariski-open in $\mathcal{W}$. Indeed, by Lemma 1.3.2 of [CM98], the fiber $\mathcal{Z}_0 \cap \pi^{-1}(\lambda)$ is empty for a given $\lambda \in \mathcal{W}$ if and only if $\omega_j \in \mathfrak{m}_\lambda$ for all j, if and only if $\mathscr{I} = (\omega_1, \omega_2, \omega_3, \dots) \subset \mathfrak{m}_\lambda$. The ideal $\mathscr{I} \subset \mathcal{O}(\mathcal{W})$ is naturally identified with the global sections of a coherent ideal sheaf over $\mathcal{W}$, which cuts out a closed analytic subset $V(\mathscr{I})$ in the usual way; the complement of $V(\mathscr{I})$ is precisely $\pi_*\mathcal{Z}_0$. Fix an arithmetic weight $\lambda_0 \in \pi_*\mathcal{Z}_0$. For some sufficiently large h_0 and some affinoid Ω_0 containing λ_0, $\mathcal{Z}_{\Omega_0,h_0}$ will contain $\mathcal{Z}_{\Omega_0,h_0} \cap \mathcal{Z}_0$ as a nonempty union of irreducible components, and the latter intersection will be finite flat over Ω_0. Since $\mathcal{M}^n(\mathcal{Z}_{\Omega_0,h_0}) \simeq H^n(K^p, \mathcal{D}_{\Omega_0})_{\leq h_0}$, we deduce that $\mathrm{Supp}_{\Omega_0} H^n(K^p, \mathcal{D}_{\Omega_0})_{\leq h_0} = \Omega_0$, whence $H^n(K^p, \mathcal{D}_{\Omega_0})_{\leq h_0}$ is faithful, so by Step One $G^{\mathrm{der}}(\mathbf{R})$ has a discrete series and $n = \frac{1}{2}\mathrm{dim} G(\mathbf{R})/Z_\infty K_\infty$.

3.4 Some cases of Urban's conjecture

In this subsection we prove Theorem 1.6.

By the basic properties of irreducible components together with the construction given in §3.2, it suffices to work locally over a fixed $\mathcal{Z}_{\Omega,h} \in \mathscr{C}\mathrm{ov}$. Suppose $\phi : \mathbf{T}_{\Omega,h} \to \overline{\mathbf{Q}_p}$ is an eigenpacket corresponding to a cuspidal non-critical regular classical point $x \in \mathscr{X}_{\Omega,h}$. Set $\mathfrak{M} = \ker \phi$, and let $\mathfrak{m} = \mathfrak{m}_\lambda$ be the contraction of $\mathfrak{M}$ to $A(\Omega)$. Let $\mathscr{P} \subset \mathbf{T}_{\Omega,h}$ be any minimal prime contained in $\mathfrak{M}$, and let $\wp$ be its contraction to a prime in $A(\Omega)$. The ring $\mathbf{T}_{\Omega,h}/\mathscr{P}$ is a finite integral extension of $A(\Omega)/\wp$, so both rings have the same dimension. In particular, Conjecture 1.5 follows from the equality $\mathrm{ht}\wp = l(G)$; this latter equality is what we prove.

Proposition 3.4.1. *The largest degrees for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ and $H^*(K^p, \mathcal{D}_\lambda)_{\leq h}$ coincide, and the smallest degrees for which ϕ occurs in $H^*(K^p, \mathcal{D}_\lambda)_{\leq h}$ and $H_*(K^p, \mathcal{A}_\Omega)_{\leq h}$ coincide. Finally, the smallest degree for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ is greater than or equal to the smallest degree for which ϕ occurs in $H_*(K^p, \mathcal{A}_\Omega)_{\leq h}$.*

Proof. For the first claim, localize the Tor spectral sequence at $\mathfrak{M}$, with $\Sigma = \lambda$. If ϕ occurs in $H^i(K^p, \mathcal{D}_\lambda)_{\leq h}$ then it occurs in a subquotient of $\mathrm{Tor}_j^{A(\Omega)}(H^{i+j}(K^p, \mathcal{D}_\Omega)_{\leq h}, A(\Omega)/\mathfrak{m}_\lambda)$ for some $j \geq 0$. On the other hand, if d is the largest degree for which ϕ occurs in $H^d(K^p, \mathcal{D}_\Omega)_{\leq h}$, the entry $E_2^{0,d}$ of the Tor spectral sequence is stable and nonzero after localizing at $\mathfrak{M}$, and it contributes to the grading on $H^d(K^p, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}}$. The second and third claims follow from an analogous treatment of the Ext spectral sequence. $\square$

First we treat the case where $l(G) = 0$, so $G^{\mathrm{der}}(\mathbf{R})$ has a discrete series. By the noncriticality of ϕ together with the nonexistence of CAP forms at regular weights and the results recalled in §3.3, the only degree for which ϕ occurs in $H^i(K^p, \mathcal{D}_\lambda)_{\leq h}$ is the middle degree $d = \frac{1}{2}\mathrm{dim} G(\mathbf{R})/Z_\infty K_\infty$, so Proposition 3.4.1 implies that the only degree for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ is the middle degree as well. The Tor spectral sequence localized at $\mathfrak{M}$ now degenerates, and yields

$$\mathrm{Tor}_i^{A(\Omega)_\mathfrak{m}}(H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}, A(\Omega)/\mathfrak{m}) = 0 \,\text{for all}\, i \geq 1.$$

By Proposition A.3, $H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$ is a *free* module over $A(\Omega)_\mathfrak{m}$, so $\mathrm{Ass}_{A(\Omega)_\mathfrak{m}}(H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}) = \{(0)\}$. Let us write $A = A(\Omega)_\mathfrak{m}$ and $\mathbf{T} = (\mathbf{T}_{\Omega,h})_\mathfrak{M}$ for brevity, so there is a diagram

$$A \to \mathbf{T} \hookrightarrow \mathrm{End}_A \left(H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}} \right).$$

I claim the structure map $A \to \mathbf{T}$ is a *flat* local homomorphism. To see this, note by the isomorphism

$$H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}} \otimes_A A/\mathfrak{m} \simeq H^d(K^p, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}}$$

that the residue ring $\mathbf{T}/\mathfrak{m}\mathbf{T}$ is simply the subalgebra of $\mathrm{End}_{A/\mathfrak{m}A} \left(H^d(K^p, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}} \right)$ generated by $\mathbf{T}(K^p)$. By the non-criticality of x together with the semisimplicity of the spherical Hecke algebra, $\mathbf{T}/\mathfrak{m}\mathbf{T}$ is a *field* and $H^d(K^p, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}}$ is a free $\mathbf{T}/\mathfrak{m}\mathbf{T}$-module of finite rank. Choosing lifts of a minimal set of $\mathbf{T}/\mathfrak{m}\mathbf{T}$-module generators of $H^d(K^p, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}}$ determines a surjection

$$\psi : \mathbf{T}^r \twoheadrightarrow H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$$

of A-modules. Applying $- \otimes_A A/\mathfrak{m}A$ to the sequence

$$0 \to \ker \psi \to \mathbf{T}^r \to H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}} \to 0$$

and using the freeness of the final term over A, we easily see that $(\ker \psi) \otimes_A A/\mathfrak{m}A = 0$, so ψ is an isomorphism. Since $H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$ is free over A and over $\mathbf{T}$, it follows that $\mathbf{T}$ is a free A-module. The rings A and $\mathbf{T}/\mathfrak{m}\mathbf{T}$ are regular, so $\mathbf{T}$ is regular by Theorem 23.7.ii of [Mat89]. Since $\mathbf{T}/\mathfrak{m}\mathbf{T}$ is a field, w is étale at x.

Now we turn to the case $l(G) \geq 1$. First, there is an affinoid open $\mathscr{Y} \subset \mathscr{X}_{\Omega,h}$ containing x, and meeting every component of $\mathscr{X}_{\Omega,h}$ containing x, such that every regular classical non-critical point in $\mathscr{Y}$ is cuspidal. Indeed, at regular weights there are no CAP representations, so the Hecke action splits the cuspidal and Eisenstein subspaces of $H^*(K^p, V_{\lambda^{\mathrm{alg}}})$, whence the top horizontal arrow in

the diagram

$$\begin{array}{ccc} H_c^i(K^p, V_{\lambda^{\mathrm{alg}}})_{\leq h} & \longrightarrow & H^i(K^p, V_{\lambda^{\mathrm{alg}}})_{\leq h} \\ \uparrow & & \uparrow \\ H_c^i(K^p, \mathcal{D}_\lambda)_{\leq h} & \longrightarrow & H^i(K^p, \mathcal{D}_\lambda)_{\leq h} \end{array}$$

becomes an isomorphism after localizing at $\mathfrak{M}$. The vertical arrows are isomorphisms by the noncriticality assumption, so the bottom arrow becomes an isomorphism after localization at $\mathfrak{M}$. Localizing the sequence

$$\cdots \to H_c^i(K^p, \mathcal{D}_\lambda) \to H^i(K^p, \mathcal{D}_\lambda) \to H_\partial^i(K^p, \mathcal{D}_\lambda) \to H_c^{i+1}(K^p, \mathcal{D}_\lambda) \to \dots$$

at $\mathfrak{M}$ then shows that ϕ does not occur in $H_\partial^*(K^p, \mathcal{D}_\lambda)_{\leq h}$, so by the boundary spectral sequence ϕ does not occur in $H_\partial^*(K^p, \mathcal{D}_\Omega)_{\leq h}$. Since $\mathrm{Supp}_{\mathbf{T}_{\Omega,h}} H_\partial^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ is closed in $\mathscr{X}_{\Omega,h}$ and does not meet x, the existence of a suitable $\mathscr{Y}$ now follows easily. Shrinking Ω and $\mathscr{Y}$ as necessary, we may assume that $A(\mathscr{Y})$ is finite over $A(\Omega)$, and thus $\mathscr{M}^*(\mathscr{Y}) = H^*(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{\mathbf{T}_{\Omega,h}} A(\mathscr{Y})$ is finite over $A(\Omega)$ as well. Exactly as in the proof of Theorem 1.2, the Tor spectral sequence shows that $\mathrm{Supp}_\Omega \mathscr{M}^*(\mathscr{Y})$ doesn't contain any regular non-self-dual weights for which h is a small slope, so $\mathscr{M}^*(\mathscr{Y})$ and $A(\mathscr{Y})$ are torsion $A(\Omega)$-modules.

Finally, suppose $l(G) = 1$. Set $d = q(G)$, so ϕ occurs in $H^*(K^p, \mathcal{D}_\lambda)_{\leq h} \simeq H^*(K^p, V_\lambda)_{\leq h}$ only in degrees d and $d+1$. By the argument of the previous paragraph and the faithful flatness of $\mathcal{O}_{\Omega,\lambda}$ over $A(\Omega)_\mathfrak{m}$, $H^*(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$ is a torsion $A(\Omega)_\mathfrak{m}$-module. Taking $\Sigma = \lambda$ in the Ext spectral sequence and localizing at $\mathfrak{M}$, Proposition 3.4.1 yields

$$H^d(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}} \simeq \mathrm{Hom}_{A(\Omega)_\mathfrak{m}}(H_d(K^p, \mathcal{A}_\Omega)_{\leq h, \mathfrak{M}}, A(\Omega)_\mathfrak{m}).$$

Since the left-hand term is a torsion $A(\Omega)_\mathfrak{m}$-module, Proposition 3.3.1 implies that both modules vanish identically. Proposition 3.4.1 now shows that $d+1$ is the only degree for which ϕ occurs in $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$. Taking $\Sigma = \lambda$ in the Tor spectral sequence and localizing at $\mathfrak{M}$, the only nonvanishing entries are $E_2^{0,d+1}$ and $E_2^{-1,d+1}$. In particular, $\mathrm{Tor}_i^{A(\Omega)_\mathfrak{m}}(H^{d+1}(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}, A(\Omega)/\mathfrak{m}) = 0$ for all $i \geq 2$, so $H^{d+1}(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$ has projective dimension at most one by Proposition A.3. Summarizing, we've shown that $H^i(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$ vanishes in degrees $\neq q(G)+1$, and that $H^{q(G)+1}(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$ is a torsion $A(\Omega)_\mathfrak{m}$-module of projective dimension one, so the theorem now follows from Propositions A.4 and A.6. □

As an exotic example of a group with $l(G) = 1$, let F be a totally real number field with $\Sigma = \{\sigma : F \hookrightarrow \mathbf{R}\}$, and choose a quadratic form Q in an even number of variables over F such that $\sigma(\mathrm{disc}(Q)) < 0$ for exactly one $\sigma \in \Sigma$. Then $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{SO}_Q$ has $l(G) = 1$.

4 The imaginary quadratic eigencurve

In this section we sketch the proof of Theorem 1.8 in the nonsplit case. The split case requires an analysis of boundary cohomology; we defer an analysis of this case to [Han12c], since that paper will develop tools for systematically calculating overconvergent cohomology over the Borel-Serre boundary.

Fix an imaginary quadratic field $F/\mathbf{Q}$ and a prime p split in F, say $p = \mathfrak{p}_1\mathfrak{p}_2$. Fix a quaternion algebra D/F, and let $\mathfrak{d} \subset \mathcal{O}_F$ be the product of the primes where D ramifies. We assume D splits

at every prime over p. Set $G = \mathrm{Res}_{F/\mathbf{Q}} D^\times$; if R is an associative $\mathbf{Q}$-algebra, we shall identify $G(R) \simeq (D \otimes_{\mathbf{Q}} R)^\times$ without particular comment. In particular, there is an isomorphism

$$\iota_1 \times \iota_2 : G(\mathbf{Q}_p) = \mathrm{GL}_2(F \otimes_{\mathbf{Q}} \mathbf{Q}_p) \simeq \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p).$$

The weight space for G is simply the product $\mathcal{W}^{(1)} \times \mathcal{W}^{(2)}$ of two copies of the weight space $\mathcal{W}$ for $\mathrm{GL}_2/\mathbf{Q}_p$. We define the *null space* $\mathcal{W}_0$ for G as the product of null subspaces $\mathcal{W}_0^{(1)} \times \mathcal{W}_0^{(2)}$, so $\mathcal{W}_0$ is two-dimensional. In particular, a weight $\lambda \in \mathcal{W}_0(L)$ is simply a pair of characters $\lambda_1, \lambda_2 : \mathbf{Z}_p^\times \to L^\times$ with $\lambda : \mathrm{diag}(x, 1) \mapsto \lambda_1(\iota_1(x))\lambda_2(\iota_2(x))$.

Given a finite place v, we write $\mathfrak{p}_v$ for the associated prime ideal of $\mathcal{O}_F$ and $\mathcal{O}_v$ for the v-adic completion of $\mathcal{O}_F$. Given two ideals $\mathfrak{m}, \mathfrak{n} \subset \mathcal{O}_F$ prime to $\mathfrak{d}$, we define an open compact subgroup $K(\mathfrak{n}, \mathfrak{m}) = \prod_v K(\mathfrak{n}, \mathfrak{m})_v$ of $G(\mathbf{A}_f)$ as follows:

- $K(\mathfrak{n}, \mathfrak{m})_v \simeq \mathrm{GL}_2(\mathcal{O}_v)$ if $\mathfrak{p}_v \nmid \mathfrak{mnd}$
- $K(\mathfrak{n}, \mathfrak{m})_v \simeq \mathcal{O}_{D_v}^\times$ if $\mathfrak{p}_v | \mathfrak{d}$, where $\mathcal{O}_{D_v}$ is a choice of maximal order in $D_v = D \otimes_F F_v$
- $K(\mathfrak{n}, \mathfrak{m})_v \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v),\, c \in (\mathfrak{m} \cap \mathfrak{n})\mathcal{O}_v,\, d - 1 \in \mathfrak{n}\mathcal{O}_v \right\}$.

This is analogous to the classical subgroup $\Gamma_1(N) \cap \Gamma_0(M)$ of $\mathrm{GL}_2(\mathbf{Z})$. The projection of $K(\mathfrak{m}, \mathfrak{n})$ onto $G(\mathbf{Q}_p)$ is contained in the Iwahori if and only if $\mathfrak{m} \cap \mathfrak{n} \subseteq (p)$. We shall be interested in the cohomology of local systems on the hyperbolic three-manifold $Y(K(\mathfrak{n}, (p)))$. For brevity, we set $H_*(\mathfrak{n}, \mathcal{A}_\Omega) = H_*(Y(K(\mathfrak{n}, (p))), \mathcal{A}_\Omega)$, etc.

For $i = 1, 2$, fix a uniformizer ϖ_i of $\mathcal{O}_{\mathfrak{p}_i}$, and let U_i be the Hecke operator associated with the matrix $\begin{pmatrix} 1 & \\ & \varpi_i \end{pmatrix}$. We set $U_p = U_1 U_2$; this is a controlling operator. Theorem 1.8 follows from the construction in §3.2 together with the following theorem.

Theorem 4.1. *Let $F/\mathbf{Q}$ be an imaginary quadratic field in which p splits. Fix a quaternion division algebra D/F split at the primes over p, and let $G/\mathbf{Q}$ be the inner form of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$ associated with D. Choose a slope datum (U_t, Ω, h) with Ω contained in the two-dimensional null space $\mathcal{W}_0 \subset \mathcal{W}$ defined in §4.3. Then $H_i(K^p, \mathcal{A}_\Omega)_{\leq h} = 0$ unless $i = 1$ and $H^i(K^p, \mathcal{D}_\Omega)_{\leq h} = 0$ unless $i = 2$ or $i = 3$. The Ext spectral sequence degenerates, for $\Sigma = \Omega$, to isomorphisms*

$$\begin{aligned} H^2(K^p, \mathcal{D}_\Omega)_{\leq h} &\simeq \mathrm{Ext}^1_{A(\Omega)}(H_1(K^p, \mathcal{A}_\Omega)_{\leq h}, A(\Omega)), \\ H^3(K^p, \mathcal{D}_\Omega)_{\leq h} &\simeq \mathrm{Ext}^2_{A(\Omega)}(H_1(K^p, \mathcal{A}_\Omega)_{\leq h}, A(\Omega)). \end{aligned}$$

For any weight $\lambda \in \Omega$, the Tor spectral sequence yields a long exact sequence

$$0 \to H^2(K^p, \mathcal{D}_\Omega)_{\leq h} \otimes_{A(\Omega)} A(\Omega)/\mathfrak{m}_\lambda \to H^2(K^p, \mathcal{D}_\lambda)_{\leq h} \to \mathrm{Tor}^1_{A(\Omega)}(H^3(K^p, \mathcal{D}_\Omega)_{\leq h}, A(\Omega)/\mathfrak{m}_\lambda) \to 0,$$

and the third term vanishes unless λ the trivial weight. Finally, $H^2(K^p, \mathcal{D}_\Omega)_{\leq h}$, if nonzero, is a Cohen-Macaulay $A(\Omega)$-module of projective dimension and grade one, and all of its associated primes have height one.

Lemma 4.3.1. *For any slope datum, we have $H^0(\mathfrak{n}, \mathcal{D}_\lambda)_{\leq h} = 0$ and $H^0(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h} = 0$.*

Proof. Let $\mathbf{D}^s$ denote the k-Banach dual of the space $\mathbf{A}^s$ of continuous k-valued functions $f : \mathbf{Z}_p \to k$ which are analytic on each coset of $p^s\mathbf{Z}_p$. Let $b \in \mathbf{Z}_p$ act on $\mathbf{A}^s$ by translation, i.e. $(t_b f)(x) = f(x + b)$, and let t_b^* denote the dual action on $\mathbf{D}^s$. By strong approximation it's enough

to show that for any $b \neq 0$, the only t_b^*-fixed element of $\mathbf{D}^s$ is the zero distribution. By Theorem 1.7.8 of [Col], the Amice transform

$$\mu \mapsto A_\mu(T) = \int_{\mathbf{Z}_p} (1+T)^x d\mu(x)$$

defines an isometric isomorphism of $\mathbf{D}^s$ onto the ring

$$\mathcal{R}^s = \left\{ \sum_{n=0}^{\infty} b_n T^n,\ b_n \in k \,\mathrm{with}\, |b_n| \ll p^{v_p(\lfloor p^{-s} n \rfloor !)} \,\mathrm{as}\, n \to \infty \right\},$$

where $\mathcal{R}^s$ is equipped with the norm $|\rho| = \sup_n |b_n(\rho)| p^{-v_p(\lfloor p^{-s} n \rfloor !)}$. The ring $\mathcal{R}^s$ is an integral domain and injects densely into the Frechet algebra of power series which converge on the open disk $|T| < p^{-\frac{1}{p^s(p-1)}}$. A simple computation gives $A_{t_b^* \mu}(T) = (1+T)^b A_\mu(T)$, so $t_b^* \mu = \mu$ implies $\left((1+T)^b - 1\right) A_\mu(T) = 0$, whence $A_\mu(T) = 0$ and $\mu = 0$ as desired. □

Proof of Theorem 4.1. Choose a slope datum (Ω, h) with $\Omega \subset \mathcal{W}_0$. By Theorem 1.2, $H_*(\mathfrak{n}, \mathcal{A}_\Omega)_{\leq h}$ and $H^*(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}$ are torsion $A(\Omega)$-modules. We examine the Tor spectral sequence for $\Sigma = \lambda \in \Omega$ an arbitrary weight; the sequence reads

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{A(\Omega)}(H^j(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}, A(\Omega)/\mathfrak{m}_\lambda) \Rightarrow H^{i+j}(\mathfrak{n}, \mathcal{D}_\lambda)_{\leq h}.$$

As the ring $A(\Omega)$ is regular of dimension two, nonzero entries on the E_2 page may only occur with coordinates $-2 \leq i \leq 0, 0 \leq j \leq 3$, and the sequence stabilizes at its E_3 page. By Lemma 4.3.1, $H^0(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h} = 0$ and so the entries $E_2^{i,0}$ along the zeroth row vanish identically. This implies that the entries $E_2^{-2,1}$ and $E_2^{-1,1}$ are stable; they contribute to gradings on $H^j(\mathfrak{n}, \mathcal{D}_\lambda)_{\leq h}$ for $j \in \{-1, 0\}$, which both vanish, so they vanish and hence $\mathrm{Tor}^i_{A(\Omega)}(H^1(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}, A(\Omega)/\mathfrak{m}_\lambda) = 0$ for any $\lambda \in \Omega$ and any $i > 0$. By Proposition A.3, $H^1(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}$ is either zero or projective, but Theorem 1.2 rules out projectivity, so $H^1(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h} = 0$. This in turn implies that the entries $E_2^{-2,2}$ and $E_2^{-1,2}$ are stable; $E_2^{-2,2}$ contributes to the grading on $H^0(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h} = 0$, so it vanishes. Thus $\mathrm{Tor}_i^{A(\Omega)}(H^2(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}, A(\Omega)/\mathfrak{m}_\lambda) = 0$ for any $\lambda \in \Omega$ and any $i > 1$, so $H^2(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}$ has projective dimension at most one by Proposition A.3; since it's already a torsion module, its grade and projective dimension are both at least one. Thus $H^2(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}$ is a perfect $A(\Omega)$-module of projective dimension one, so the final sentence of Theorem 4.1 follows from Proposition A.6.

An analysis as in Lemma 3.9 of [Bel12] shows that $H^3(\mathfrak{n}, \mathcal{D}_\Omega)$ is zero unless Ω contains the trivial weight λ_0, in which case there is an $A(\Omega)$-module isomorphism $H^3(\mathfrak{n}, \mathcal{D}_\Omega) \simeq (A(\Omega)/\mathfrak{m}_{\lambda_0})^{|\mathrm{Cl}(F)|}$. A simple calculation shows $H_3(\mathfrak{n}, \mathcal{A}_\Omega) = 0$, so the remainder of the theorem will follow if we can show $H_2(\mathfrak{n}, \mathcal{A}_\Omega)_{\leq h} = 0$. Considering the Ext spectral sequence with $\Omega = \lambda$, the entry $E_2^{2,2}$ is stable and necessarily vanishes for any $\lambda \in \Omega$, so $\mathrm{projdim}_{A(\Omega)}(H_2(\mathfrak{n}, \mathcal{A}_\Omega)_{\leq h}) \leq 1$. The entry $E_2^{1,2}$ is stable for any λ, and contributes to the grading on $H^3(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h}$, which has zero dimensional support, so $E_2^{1,2}$ must vanish for λ outside a zero-dimensional analytic set. This rules out the possibility that $H_2(\mathfrak{n}, \mathcal{A}_\Omega)_{\leq h}$ has projective dimension one, and Theorem 1.2 rules out projectivity. □

Two conjectures

Fix a character $\lambda \in \mathcal{W}_0$ with value field k. Note that λ is simply a pair of characters λ_1, λ_2 with $\lambda_i : \mathbf{Z}_p^\times \to k^\times$. Set $\delta(\lambda_i) = \frac{\log \lambda_i(1+p)}{\log(1+p)} \in k$.

For each prime $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}$, choose a uniformizer $\varpi_{\mathfrak{l}}$ of $\mathcal{O}_{v_{\mathfrak{l}}}$ and let $T_{\mathfrak{l}}$ be the double coset operator associated with the matrix $\begin{pmatrix} 1 & \\ & \varpi_{\mathfrak{l}} \end{pmatrix}$. For $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}p$, let $S_{\mathfrak{l}}$ be the double coset operator of $\begin{pmatrix} \varpi_{\mathfrak{l}} & \\ & \varpi_{\mathfrak{l}} \end{pmatrix}$. We define the $\mathfrak{n}$-new subspace $H^2(\mathfrak{n}, \mathcal{D}_\lambda)^{\mathfrak{n}-\mathrm{new}}$ of $H^2(\mathfrak{n}, \mathcal{D}_\lambda)$ by intersecting the kernels of the various level-lowering maps into $H^2(\mathfrak{n}', \mathcal{D}_\lambda)^{\oplus 2}$ for varying $\mathfrak{n}'|\mathfrak{n}$ in the usual way. Write $\mathbf{T}^{\mathrm{new}}_\lambda(\mathfrak{n})$ for subalgebra of $\mathrm{End}_k(H^2(\mathfrak{n}, \mathcal{D}_\lambda)_{<\infty})$ generated by $T_{\mathfrak{l}}$ for all $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}$ and by $S_{\mathfrak{l}}$ for all $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}p$.

Conjecture. *Given a k-algebra homomorphism $\pi : \mathbf{T}^{\mathrm{new}}_\lambda(\mathfrak{n}) \to \mathbf{C}_p$, there is a continuous semisimple Galois representation $\rho_\pi : G_F \to \mathrm{GL}_2(\mathbf{C}_p)$, unique up to conjugation, satisfying the following properties:*

i. For any $\mathfrak{l} \nmid \mathfrak{n}p$, $\mathrm{tr}\rho_\pi(\mathrm{Frob}_{\mathfrak{l}}) = \pi(T_{\mathfrak{l}})$ and $\det\rho(\mathrm{Frob}_{\mathfrak{l}}) = \mathrm{Nm}(\mathfrak{l})\pi(S_{\mathfrak{l}})$.

ii. The prime-to-p Artin conductor of ρ_π divides $\mathfrak{n}\mathfrak{d}$.

iii. The Hodge-Tate-Sen weights of ρ_π at the decomposition groups over p are $\{0, \delta(\lambda_1)+1\}$ and $\{0, \delta(\lambda_2)+1\}$.

iv. The representation $\rho_\pi|D_{\mathfrak{p}}$ is trianguline for all $\mathfrak{p}|p$.

Next, recall that our group G is the inner form of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$ associated with the quaternion algebra of discriminant $\mathfrak{d}$, where $\mathfrak{d}$ is a squarefree ideal prime to p and divisible by an even number of prime ideals. Given a slope datum, let $\mathbf{T}^{\mathfrak{d}}_{\Omega,h}(\mathfrak{n})$ denote the subalgebra of $\mathrm{End}_{A(\Omega)}(H^2(\mathfrak{n}, \mathcal{D}_\Omega)_{\leq h})$ generated by $T_{\mathfrak{l}}$ for all $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}$ and by $S_{\mathfrak{l}}$ for all $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}p$. We wish to formulate a Jacquet-Langlands conjecture relating the algebra $\mathbf{T}^{\mathfrak{d}}_{\Omega,h}(\mathfrak{n})$ with an analogous algebra defined using the split form of G.

Let $G' = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$ be the split form, and let $K(\mathfrak{n}, \mathfrak{d})$ denote the subgroup of $G^{'}(\mathbf{A}_f)$ defined as above (so the second bulleted condition is vacuous). Let $H^2_!(K(\mathfrak{n}, p\mathfrak{d}), \mathcal{D}_\Omega)^{\mathfrak{d}-\mathrm{new}}_{\leq h}$ denote the intersection of the parabolic cohomology with the kernels of the various level-lowering maps, and let $\mathbf{T}_{\Omega,h}(\mathfrak{n}, \mathfrak{d})^{\mathfrak{d}-\mathrm{new}}_!$ denote the subalgebra of

$$\mathrm{End}_{A(\Omega)}\left(H^2_!(K(\mathfrak{n}, p\mathfrak{d}), \mathcal{D}_\Omega)^{\mathfrak{d}-\mathrm{new}}_{\leq h}\right)$$

generated by $T_{\mathfrak{l}}$ for all $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}$ and by $S_{\mathfrak{l}}$ for all $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{d}p$.

Conjecture. *There is an $A(\Omega)$-algebra isomorphism*

$$\mathbf{T}^{\mathfrak{d}}_{\Omega,h}(\mathfrak{n}) \simeq \mathbf{T}_{\Omega,h}(\mathfrak{n}, \mathfrak{d})^{\mathfrak{d}-\mathrm{new}}_!.$$

A Some commutative algebra

In this appendix we collect some results relating the projective dimension of a module M and its localizations, the nonvanishing of certain Tor and Ext groups, and the heights of the associated primes of M. We also briefly recall the definition of a perfect module, and explain their basic properties. These results are presumably well-known to experts, but they are not given in our basic reference [Mat89].

Throughout this subsection, R is a commutative Noetherian ring and M is a finite R-module. Our notations follow [Mat89], with one addition: we write $\mathrm{mSupp}(M)$ for the set of maximal ideals in $\mathrm{Supp}(M)$.

Proposition A.1. *There is an equivalence*

$$\mathrm{projdim}_R(M) \geq n \Leftrightarrow \mathrm{Ext}^n_R(M, N) \neq 0 \,\text{for some}\, N \in \mathrm{Mod}_R.$$

See e.g. p. 280 of [Mat89] for a proof.

Proposition A.2. *The equality*

$$\mathrm{projdim}_R(M) = \sup_{\mathfrak{m}\in \mathrm{mSupp}(M)} \mathrm{projdim}_{R_\mathfrak{m}}(M_\mathfrak{m})$$

holds.

Proof. Any projective resolution of M localizes to a projective resolution of $M_\mathfrak{m}$, so $\mathrm{projdim}_{R_\mathfrak{m}}(M_\mathfrak{m}) \leq \mathrm{projdim}_R(M)$ for all $\mathfrak{m}$. On the other hand, if $\mathrm{projdim}_R(M) \geq n$, then $\mathrm{Ext}^n_R(M,N) \neq 0$ for some N, so $\mathrm{Ext}^n_R(M,N)_\mathfrak{m} \neq 0$ for some $\mathfrak{m}$; but $\mathrm{Ext}^n_R(M,N)_\mathfrak{m} \simeq \mathrm{Ext}^n_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$, so $\mathrm{projdim}_{R_\mathfrak{m}}(M_\mathfrak{m}) \geq n$ for some $\mathfrak{m}$ by Proposition A.1. □

Proposition A.3. *For M any finite R-module, the equality*

$$\mathrm{projdim}_R(M) = \sup_{\mathfrak{m}\in \mathrm{mSupp}(M)} \sup\left\{i | \mathrm{Tor}^R_i(M, R/\mathfrak{m}) \neq 0\right\}$$

holds. If furthermore $\mathrm{projdim}_R(M) < \infty$ *then the equality*

$$\mathrm{projdim}_R(M) = \sup\left\{i | \mathrm{Ext}^i_R(M,R) \neq 0\right\}$$

holds as well.

Proof. The module $\mathrm{Tor}^R_i(M, R/\mathfrak{m})$ is a finite-dimensional $R/\mathfrak{m}$-vector space, so localization at $\mathfrak{m}$ leaves it unchanged, yielding

$$\begin{aligned} \mathrm{Tor}^R_i(M, R/\mathfrak{m}) &\simeq \mathrm{Tor}^R_i(M, R/\mathfrak{m})_\mathfrak{m} \\ &\simeq \mathrm{Tor}^{R_\mathfrak{m}}_i(M_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{m}). \end{aligned}$$

Since the equality $\mathrm{projdim}_S(N) = \sup\left\{i | \mathrm{Tor}^S_i(N, S/\mathfrak{m}_S) \neq 0\right\}$ holds for any local ring S and any finite S-module N (see e.g. Lemma 19.1.ii of [Mat89]), the first claim now follows from Proposition A.2.

For the second claim, we first note that if S is a local ring and N is a finite S-module with $\mathrm{projdim}_S(N) < \infty$, then $\mathrm{projdim}_S(N) = \sup\{i | \mathrm{Ext}^i_S(N,S) \neq 0\}$ by Lemma 19.1.iii of [Mat89]. Hence by Proposition A.2 we have

$$\begin{aligned} \mathrm{projdim}_R(M) &= \sup_{\mathfrak{m}\in \mathrm{mSupp}(M)} \sup\left\{i | \mathrm{Ext}^i_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}) \neq 0\right\} \\ &= \sup\{i | \mathrm{Ext}^i_R(M,R)_\mathfrak{m} \neq 0 \,\mathrm{for\,some}\, \mathfrak{m}\} \\ &= \sup\{i | \mathrm{Ext}^i_R(M,R) \neq 0\}, \end{aligned}$$

as desired. □

Proposition A.4. *If R is a Cohen-Macaulay ring, M is a finite R-module of finite projective dimension, and $\mathfrak{p}$ is an associated prime of M, then* $\mathrm{ht}\mathfrak{p} = \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p})$. *In particular,* $\mathrm{ht}\mathfrak{p} \leq \mathrm{projdim}_R(M)$.

Proof. Supposing $\mathfrak{p}$ is an associated prime of M, there is an injection $R/\mathfrak{p} \hookrightarrow M$; this localizes to an injection $R_\mathfrak{p}/\mathfrak{p} \hookrightarrow M_\mathfrak{p}$, so $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$. Now we compute

$$\begin{aligned} \mathrm{ht}\mathfrak{p} &= \dim(R_\mathfrak{p}) \\ &= \mathrm{depth}_{R_\mathfrak{p}}(R_\mathfrak{p}) \,(\mathrm{by\,the\,CM\,assumption}) \\ &= \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \,(\mathrm{by\,the\,Auslander-Buchsbaum\,formula}) \\ &= \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}), \end{aligned}$$

whence the result. □

Now we single out an especially nice class of modules, which are equidimensional in essentially every sense of the word. Recall the *grade* of a module M, written $\mathrm{grade}_R(M)$, is the $\mathrm{ann}_R(M)$-depth of R; by Theorems 16.6 and 16.7 of [Mat89],

$$\mathrm{grade}_R(M) = \inf\{i | \mathrm{Ext}^i_R(M, R) \neq 0\},$$

so quite generally $\mathrm{grade}_R(M) \leq \mathrm{projdim}_R(M)$.

Definition A.5. *A finite R-module M is* perfect *if* $\mathrm{grade}_R(M) = \mathrm{projdim}_R(M) < \infty$.

Proposition A.6. *Let R be a Noetherian ring, and let M be a perfect R-module, with* $\mathrm{grade}_R(M) = \mathrm{projdim}_R(M) = d$. *Then for any $\mathfrak{p} \in \mathrm{Supp}(M)$ we have* $\mathrm{grade}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = d$. *If furthermore R is Cohen-Macaulay, then M is Cohen-Macaulay as well, and every associated prime of M has height d.*

Proof. The grade of a module can only increase under localization (as evidenced by the Ext definition above), while the projective dimension can only decrease; on the other hand, $\mathrm{grade}_R(M) \leq \mathrm{projdim}_R(M)$ for any finite module over any Noetherian ring. This proves the first claim.

For the second claim, Theorems 16.6 and 17.4.i of [Mat89] combine to yield the formula

$$\dim(M_\mathfrak{p}) + \mathrm{grade}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim(R_\mathfrak{p})$$

for any $\mathfrak{p} \in \mathrm{Supp}(M)$. The Auslander-Buchsbaum formula reads

$$\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(R_\mathfrak{p}).$$

But $\dim(R_\mathfrak{p}) = \mathrm{depth}_{R_\mathfrak{p}}(R_\mathfrak{p})$ by the Cohen-Macaulay assumption, and $\mathrm{grade}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{projdim}_{R_\mathfrak{p}}(M_\mathfrak{p})$ by the first claim. Hence $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim(M_\mathfrak{p})$ as desired. The assertion regarding associated primes is immediate from the first claim and Proposition A.4. □

B The dimension of regular cuspidal components

by James Newton[6]

In this appendix we use the results of the above article to give some additional evidence for Conjecture 1.5. In the notation and terminology of Section 1 above, we prove

Proposition B.1. *Any irreducible component of $\mathscr{X}(K^p)$ containing a cuspidal non-critical regular classical point has dimension at least* $\dim(\mathcal{W}) - l(G)$.

Note that Proposition 5.7.4 of [Urb11] implies that at least one of these components has dimension at least $\dim(\mathcal{W}) - l(G)$. This is stated without proof in that reference, and is due to G. Stevens and E. Urban. We learned the idea of the proof of this result from E. Urban — in this appendix we adapt that idea and make essential use of Theorem 1.1 (in particular the 'Tor spectral sequence') to provide a fairly simple proof of Proposition B.1.

We place ourselves in the setting of Section 1. In particular, G is a reductive group over $\mathbf{Q}$, which is split over $\mathbf{Q}_p$. Fix an open compact subgroup $K^p \subset G(\mathbf{A}_f^p)$ and a slope datum (U_t, Ω, h). Set $q = q(G)$, $l = l(G)$ (these quantities are defined in the paragraph before the statement of Conjecture 1.5), and suppose that $\mathfrak{M}$ is a maximal ideal of $\mathbf{T}_{\Omega,h}(K^p)$ corresponding to a cuspidal non-critical

[6]Trinity College, Cambridge, CB2 1TQ; jjmn2@cam.ac.uk

regular classical point of $\mathscr{X}(K^p)$. Denote by $\mathfrak{m}$ the contraction of $\mathfrak{M}$ to $A(\Omega)$. Let $\mathscr{P}$ be a minimal prime of $\mathbf{T}_{\Omega,h}(K^p)$ contained in $\mathfrak{M}$. Since $H^*(K^p, \mathcal{D}_\Omega)_{\leq h}$ is a finite faithful $\mathbf{T}_{\Omega,h}(K^p)$-module, minimal primes of $\mathbf{T}_{\Omega,h}(K^p)$ are in bijection with minimal elements of

$$\mathrm{Supp}_{\mathbf{T}_{\Omega,h}(K^p)}(H^*(K^p, \mathcal{D}_\Omega)_{\leq h});$$

by Theorem 6.5 of [Mat89], minimal elements of the latter set are in bijection with minimal elements of

$$\mathrm{Ass}_{\mathbf{T}_{\Omega,h}(K^p)}(H^*(K^p, \mathcal{D}_\Omega)_{\leq h}).$$

Definition B.2. *Denote by r the minimal index i such that $\mathscr{P}$ is in the support of $H^i(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$.*

Let $\wp$ denote the contraction of $\mathscr{P}$ to a prime of $A(\Omega)_\mathfrak{m}$; in particular, $\wp$ is an associated prime of $H^r(K^p, \mathcal{D}_\Omega)_{\leq h, \mathfrak{M}}$. The ring $A(\Omega)_\mathfrak{m}$ is a regular local ring. The localisation $A(\Omega)_\wp$ is therefore a regular local ring, with maximal ideal $\wp A(\Omega)_\wp$. We let $(x_1, ..., x_d)$ denote a regular sequence generating $\wp A(\Omega)_\wp$. After multiplying the x_i by units in $A(\Omega)_\wp$, we may assume that the x_i are in $A(\Omega)$. Note that $(x_1, ..., x_d)A(\Omega)_\mathfrak{m}$ may be a proper submodule of $\wp$. Nevertheless, we have

$$d = \dim(A(\Omega)_\wp) = \mathrm{ht}(\wp).$$

We will show that $d \leq l$.

Denote by A_i the quotient $A(\Omega)_\wp/(x_1, ..., x_i)A(\Omega)_\wp$ and denote by Σ_i the Zariski closed subspace of Ω defined by the ideal $(x_1, ..., x_i)A(\Omega)$. The affinoids Σ_i may be non-reduced. Note that $A_i = A(\Sigma_i)_\wp$ and $A(\Sigma_{i+1}) = A(\Sigma_i)/x_{i+1}A(\Sigma_i)$.

Lemma B.3. *The space*

$$H^{r-d}(K^p, \mathcal{D}_{\Sigma_d})_{\leq h, \mathscr{P}}$$

is non-zero.

Proof. By induction, it suffices to prove the following: let i be an integer satisfying $0 \leq i \leq d-1$. Suppose

$$H^{r-i}(K^p, \mathcal{D}_{\Sigma_i})_{\leq h, \mathscr{P}}$$

is a non-zero A_i-module, with $\wp A_i$ an associated prime, and

$$H^t(K^p, \mathcal{D}_{\Sigma_i})_{\leq h, \mathscr{P}} = 0$$

for every $t < r - i$. Then

$$H^{r-i-1}(K^p, \mathcal{D}_{\Sigma_{i+1}})_{\leq h, \mathscr{P}}$$

is a non-zero A_{i+1}-module, with $\wp A_{i+1}$ an associated prime, and

$$H^t(K^p, \mathcal{D}_{\Sigma_{i+1}})_{\leq h, \mathscr{P}} = 0$$

for every $t < r - i - 1$.

Note that the hypothesis of this claim holds for $i = 0$, by the minimality of r. Suppose the hypothesis is satisfied for i. It will suffice to show that

- $H^t(K^p, \mathcal{D}_{\Sigma_{i+1}})_{\leq h, \mathscr{P}} = 0$ for every $t < r - i - 1$

- there is an isomorphism of non-zero A_i-modules

$$\iota : \mathrm{Tor}_1^{A_i}(H^{r-i}(K^p, \mathcal{D}_{\Sigma_i})_{\leq h, \mathscr{P}}, A_i/x_{i+1}A_i) \cong H^{r-i-1}(K^p, \mathcal{D}_{\Sigma_{i+1}})_{\leq h, \mathscr{P}}.$$

Indeed, the left hand side (which we denote by T) of the isomorphism ι is given by the x_{i+1}-torsion in $H^{r-i}(K^p, \mathcal{D}_{\Sigma_i})_{\leq h, \mathscr{P}}$, so a non-zero A_i-submodule of $H^{r-i}(K^p, \mathcal{D}_{\Sigma_i})_{\leq h, \mathscr{P}}$ with annihilator $\wp A_i$ immediately gives a non-zero A_{i+1}-submodule of T with annihilator $\wp A_{i+1}$.

Both the claimed facts are shown by studying the localisation at $\mathscr{P}$ of the spectral sequence

$$E_2^{s,t} : \mathrm{Tor}_{-s}^{A(\Sigma_i)}(H^t(K^p, \mathcal{D}_{\Sigma_i})_{\leq h}, A(\Sigma_{i+1})) \Rightarrow H^{s+t}(K^p, \mathcal{D}_{\Sigma_{i+1}})_{\leq h}$$

(cf. Remark 3.1.1). After localisation at $\mathscr{P}$, the spectral sequence degenerates at E_2. This is because we have a free resolution

$$0 \to A_i \overset{\times x_{i+1}}{\to} A_i \to A_{i+1} \to 0$$

of A_{i+1} as an A_i-module (we use the fact that x_{i+1} is not a zero-divisor in A_i), so $(E_2^{s,t})_{\mathscr{P}}$ vanishes whenever $s \notin \{-1, 0\}$. Moreover, since

$$H^t(K^p, \mathcal{D}_{\Sigma_i})_{\leq h, \mathscr{P}} = 0$$

for every $t < r-i$, we know that $(E_2^{s,t})_{\mathscr{P}}$ vanishes for $t < r-i$. The existence of the isomorphism ι and the desired vanishing of $H^t(K^p, \mathcal{D}_{\Sigma_{i+1}})_{\leq h, \mathscr{P}}$ are therefore demonstrated by the spectral sequence, since the only non-zero term $(E_2^{s,t})_{\mathscr{P}}$ contributing to $(E_\infty^{r-i-1})_{\mathscr{P}}$ is given by $s = -1, t = r-i$, whilst $(E_2^{s,t})_{\mathscr{P}} = 0$ for all (s,t) with $s+t < r-i-1$. □

Corollary B.4. *We have an inequality $r - d \geq q$. Since $r \leq q + l$ we obtain $d \leq l$. In particular $\wp$ has height $\leq l$, so the irreducible component of $\mathbf{T}_{\Omega,h}(K^p)$ corresponding to $\mathscr{P}$ has dimension $\geq \dim(\Omega) - l$.*

Proof. It follows from Proposition 3.4.1 (with Ω replaced by Σ_d) that

$$H^i(K^p, \mathcal{D}_{\Sigma_d})_{\leq h, \mathfrak{M}}$$

is zero for $i < q$. Our Lemma therefore implies that $r - d \geq q$. The conclusion on dimensions follows from the observation made in Section 3.4 that $\mathbf{T}_{\Omega,h}(K^p)/\mathscr{P}$ has the same dimension as $A(\Omega)/\wp$. □

Proposition B.1 follows immediately from the Corollary. We have also shown that if $d = l$ (which would be the case if Conjecture 1.5 is true), then $r = q + l$.

References

[APS08] Avner Ash, David Pollack, and Glenn Stevens. Rigidity of p-adic cohomology classes of congruence subgroups of $\mathrm{GL}(n, \mathbb{Z})$. *Proc. Lond. Math. Soc. (3)*, 96(2):367–388, 2008.

[AS08] Avner Ash and Glenn Stevens. p-adic deformations of arithmetic cohomology. 2008. preprint.

[BC04] Joël Bellaïche and Gaëtan Chenevier. Formes non tempérées pour U(3) et conjectures de Bloch-Kato. *Ann. Sci. École Norm. Sup. (4)*, 37(4):611–662, 2004.

[BC09] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, (324):xii+314, 2009.

[BC11] Joël Bellaïche and Gaëtan Chenevier. The sign of Galois representations attached to automorphic forms for unitary groups. *Compos. Math.*, 147(5):1337–1352, 2011.

[Bel12] Joël Bellaïche. Critical p-adic L-functions. *Invent. Math.*, 189(1):1–60, 2012.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.

[BLGGT12] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Local-global compatibility for l=p, ii. *Ann. Sci. de l'ENS*, to appear, 2012.

[BS73] A. Borel and J.-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973. Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.

[Buz07] Kevin Buzzard. Eigenvarieties. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 59–120. Cambridge Univ. Press, Cambridge, 2007.

[BW00] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[CE12] Frank Calegari and Matthew Emerton. Completed cohomology—a survey. In *Non-abelian fundamental groups and Iwasawa theory*, volume 393 of *London Math. Soc. Lecture Note Ser.*, pages 239–257. Cambridge Univ. Press, Cambridge, 2012.

[Che04] Gaëtan Chenevier. Familles p-adiques de formes automorphes pour GL_n. *J. Reine Angew. Math.*, 570:143–217, 2004.

[Che05] Gaëtan Chenevier. Une correspondance de Jacquet-Langlands p-adique. *Duke Math. J.*, 126(1):161–194, 2005.

[CM98] R. Coleman and B. Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.

[Col96] Robert F. Coleman. Classical and overconvergent modular forms. *Invent. Math.*, 124(1-3):215–241, 1996.

[Col97] Robert F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997.

[Con99] Brian Conrad. Irreducible components of rigid spaces. *Ann. Inst. Fourier (Grenoble)*, 49(2):473–541, 1999.

[Con06] Brian Conrad. Relative ampleness in rigid geometry. *Ann. Inst. Fourier (Grenoble)*, 56(4):1049–1126, 2006.

[Eme06] Matthew Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.

[Han12a] David Hansen. Completed and overconvergent cohomology: a comparison. 2012. in preparation.

[Han12b] David Hansen. New tools and results for p-adic automorphic cohomology. 2012. Ph.D. thesis, in preparation.

[Han12c] David Hansen. Overconvergent cohomology along the borel-serre boundary, and stratifications of eigenvarieties. 2012. in preparation.

[Hid86] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.

[Hid88] Haruzo Hida. On p-adic Hecke algebras for GL_2 over totally real fields. *Ann. of Math. (2)*, 128(2):295–384, 1988.

[Hid94] Haruzo Hida. p-adic ordinary Hecke algebras for GL(2). *Ann. Inst. Fourier (Grenoble)*, 44(5):1289–1322, 1994.

[Kat73] Nicholas M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350. Springer, Berlin, 1973.

[LS04] Jian-Shu Li and Joachim Schwermer. On the Eisenstein cohomology of arithmetic groups. *Duke Math. J.*, 123(1):141–169, 2004.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Ser73] Jean-Pierre Serre. Formes modulaires et fonctions zêta p-adiques. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972)*, pages 191–268. Lecture Notes in Math., Vol. 350. Springer, Berlin, 1973.

[Ski09] Christopher Skinner. A note on the p-adic Galois representations attached to Hilbert modular forms. *Doc. Math.*, 14:241–258, 2009.

[ST03] Peter Schneider and Jeremy Teitelbaum. Algebras of p-adic distributions and admissible representations. *Invent. Math.*, 153(1):145–196, 2003.

[Ste94] Glenn Stevens. Rigid analytic modular symbols. 1994. unpublished, 1994.

[Ste00] Glenn Stevens. Families of overconvergent modular symbols. 2000. unpublished notes for a course at Institute Henri Poicare, 2000.

[Urb11] Eric Urban. Eigenvarieties for reductive groups. *Ann. of Math. (2)*, 174(3):1685–1784, 2011.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Wil90] A. Wiles. The Iwasawa conjecture for totally real fields. *Ann. of Math. (2)*, 131(3):493–540, 1990.